Optimal Public Good Provision and Taxation with Heterogeneous Risk Preferences[*]

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Abstract

This paper develops a theory of optimal public good provision and taxation when individuals differ by their labour productivities and risk preferences. Under private provision of public goods, there is an inefficient allocation of aggregate risk between private and public consumption, as individuals fail to internalize the insurance spillovers provided to agents with different risk attitudes. I then derive sufficient-statistic tax formulas under public provision. The optimal non-linear labour tax and the riskless return tax redistribute, respectively, across income levels and across risk aversion levels conditional on income, while internalizing fiscal externalities on capital tax bases. Both the progressivity of the labour income tax schedule and the sign of the riskless return tax depend on the joint distribution of income and risk preferences, and on the welfare weights used to compare agents with different risk preferences. A linear tax on risky excess returns plays an insurance role, balancing the different tastes for risk at the societal level. Finally, I characterize an alternative non-linear tax schedule on expected capital income: distorting portfolio choices can provide an additional margin for screening unobserved risk preferences.

JEL classification: H21, H41.

Keywords: public good provision, heterogeneous risk preferences, optimal taxation, insurance, equity.

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1 Introduction

Individual attitudes toward risk play an important role in shaping both personal and collective choices. People with different attitudes to risk not only make different individual decisions throughout their lives, but also have different views on the way governments should handle risk while providing public goods. For instance, infrastructure investments are perceived by some people as too risky, either because these projects might take longer to be completed, or because the net benefits of the projects might fail to materialise.[1] Similarly, attitudes towards environmental policies are heterogeneous. Most people agree climate change is a problem and measures must be taken to mitigate or stop it, but different views on the scale of the actions to implement emerge.[2] Alternatively, public budgets might be themselves risky, as in the case of local and regional governments which often rely on substantial funding coming from the state/federal level, and are therefore subject to economic downturns. Similarly, funding of sovereign wealth funds can be affected by negative shocks in the financial markets. When public goods are financed by such volatile revenue streams, individuals with different risk attitudes will disagree on how the government should balance public investments against revenue risk.

There is now increasing evidence of heterogeneous risk preferences in the population (Bach et al., 2020; Fagereng et al., 2020; Falk et al., 2018; Von Gaudecker et al., 2011), of how these preferences change over the course of life (Dohmen et al., 2017), and of how they correlate with other individual characteristics such as cognitive ability (Dohmen et al., 2010, 2018). However, the normative issue of public provision of public goods in the presence of heterogeneous risk preferences, and the optimal tax system needed to implement, has not received much attention. This paper addresses this question by providing optimality conditions and tax formulas in terms of appropriately defined elasticities and social marginal welfare weights, highlighting the

[1]For instance, a survey conducted in 2018 on the high speed rail project "HS2", that is currently under construction in the UK, shows that 59% of those who oppose the project are worried that "costs are or potentially will be too high", possibly reflecting different preferences for risk associated with skyrocketing costs of public projects.

[2]While differences across countries can be explained by different exposures to climate risks (Dechezleprêtre et al., 2025), differences within countries could partly reflect different attitudes to (climate change) risk.

trade-off in this economy: redistributing across income and risk aversion levels at the cost of distorting labour supply, savings and portfolio decisions.

In the first part of the paper (sections 2-3), I consider a two-period framework in which agents differ only by their risk preferences, and labour supply is exogenous. I analyse the Nash equilibrium outcome when the public good is offered on the basis of individual voluntary contributions, and the solution of a social planner maximising a weighted sum of utilities, from which a risk-adjusted Samuelson rule is derived (section 2). Section 3 introduces distortionary taxation in this simplified environment: the public good is financed by linear taxes on savings. I study the benchmark case in which the government observes risk preferences and can therefore levy type-specific tax rates, and the second-best case in which a uniform rate applies to all agents.

Then, in section 4, I study the role of labour and capital income taxes to finance a public good in a more complex environment. Agent-types are characterised by their level of risk aversion and labour productivity (wage), and make decisions on labour supply, consumption and savings, with constant relative risk aversion utility. They also make portfolio decisions, choosing between two types of assets: one is risk-free, while the other is subject to aggregate risk, but with a positive expected excess return.[3] In this environment, agents with equal labour productivities but different risk preferences make different labour supply, savings and portfolio decisions, which is why the zero capital income tax result does not hold (Atkinson and Stiglitz, 1976; Saez, 2002). The government raises revenues with a non-linear tax schedule on labour earnings and with two distinct proportional taxes on the riskless (normal) return and the risky excess return (with symmetric loss offsets). As an alternative to the linear taxes on capital income, non linear taxation of expected capital income is also considered. Individual utilities are aggregated through a set of welfare weights that make them interpersonally comparable (Grant et al., 2010).[4]

When agents differ along risk preferences as well as individual productivity, public good provision and taxes play two main roles. First, it is a way to insure agents against

[3]Following the optimal tax literature, I distinguish two different components of the rate of return on savings: the (riskless) normal return and the excess return. The normal return is the price for forgoing present-time-consumption. The excess return reflects the presence of aggregate risk in the economy and drives returns heterogeneity.

[4]Additional info is provided in section 4.3.

aggregate risk. The public good allows individuals to shift risk from private to public consumption, as in the work of Christiansen (1993), Schindler (2008) and Boadway and Spiritus (2024). The public good is itself risky since tax revenues depend on the state of the economy. Second, the public good serve a redistributive role, both across income levels, and across risk aversion levels conditional on income. While the tax parameters are chosen optimally ex-ante, the policy is implemented in the second period, after the realization of the state of the economy, such that the budget is balanced. This captures the idea that funds for a specific project might be limited in the short run, so that the project gets paused or reduced in scale if a negative shock occurs.[5]

This paper presents three main sets of findings. First, the market would generally provide an inefficient probability distribution of the public good in the case of heterogeneous risk preferences. Beside the standard under-provision result, I show that the Nash equilibrium entails an allocation of risk that is welfare-inferior relative to the first best social planner solution. This is because agents do not internalise the fact that their contribution to the public good affects the distribution of risk between private and public consumption also for the other agents who have different risk preferences. Hence, agents underestimate the insurance effect that the public good provides to the other individuals. It is therefore possible to increase welfare by reshuffling the contribution rates of different types of agent. This inefficiency remains under public provision if the government cannot observe risk preferences: a uniform linear tax on savings delivers both a welfare-inferior allocation of risk and a lower expected level of public good provision, relative to the benchmark in which risk-aversion types can be taxed at type-specific rates.

Second, I analyse the optimal tax system in the case in which the public good is financed by a non-linear tax schedule on labour earnings and taxation of capital income. The non-linear labour income tax schedule, $T(z)$, performs the standard redistributive role in the Mirrleesian environment (Mirrlees, 1971; Saez, 2001): it redistributes from high- to low-income individuals according to the average social

[5] Alternatively, if we consider the environment as the public good, the risk in the economy is given by the climate risk associated with the range of possible future scenarios.

marginal welfare weight at each income level, while internalising the fiscal externalities its behavioural responses impose on capital income tax revenue. The linear tax on the riskless return, τ_r, complements the role of the labour income tax by redistributing across risk aversion types, within the same income level. Conditional on income, more risk-averse agents save less, so the sign of the optimal rate depends on how the social marginal welfare weights vary with risk aversion within income levels. Both the degree of progressivity of the labour income tax schedule, as well as the sign of the tax rate on the normal return depend on the joint distribution of income and risk preferences, as well as individual welfare weights.[6] The linear tax on the risky excess return, τ_k, by contrast, plays a purely insurance-related function, aligning the volatility of the publicly provided good with a weighted average of private consumption volatility across risk-aversion types. Indeed, individuals with different preferences for risk have different benefits from the tax/insurance policy at the margin, meaning their willingness to shift risk from the individual to the societal level depends on their risk aversion. While it is essential for the allocation of aggregate risk, the excess-return tax cannot be used to redistribute across productivity or risk-aversion types.

Third, I also characterise the optimal non-linear taxation of capital income, $T_k(\bar{y})$, evaluated on the basis of each household's expected capital income. Basing the schedule on expected, rather than realised, capital income allows the government to commit to an ex-ante schedule even though returns are realised only in the second period. This instrument can in principle screen on risk aversion beyond what the linear riskless-return tax already achieves, since expected capital income depends on risk aversion through both the level of savings and the optimally chosen portfolio share. However, this additional screening margin comes at the cost of distorting both the savings and portfolio composition. The nonlinear instrument is more likely to improve on the linear pair specifically when the within-income variation in welfare weights across risk-aversion types is driven substantially by portfolio decisions rather than savings levels alone, which in turn requires risk preferences to be widely dispersed within, and weakly correlated with, income levels.

[6] Whether the planner should favour more or less risk-averse individuals, conditional on income, is ultimately a normative question about how social welfare treats preference heterogeneity.

This paper contributes to two strands of the literature. The first concerns public good provision in a risky environment. Christiansen (1993), Schindler (2008), and Boadway and Spiritus (2024) study the role of capital income taxation in allocating risk between private and public consumption under aggregate uncertainty, with Christiansen (1993) already extending the Samuelson rule to a setting with aggregate risk. The first contribution of this paper is introducing heterogeneity in risk preferences across agents, both under private provision of the public good and under public provision financed by taxes. Under private provision, I show that the standard free-riding underprovision result is accompanied by an additional inefficiency: the Nash equilibrium also misallocates risk between private and public consumption, because each contributor's risk-bearing decision provides an insurance benefit to other agents with different risk preferences that they do not internalise, leaving the equilibrium risk allocation inferior to the first-best. Under public provision, the key novelty relative to Christiansen (1993), Schindler (2008), and Boadway and Spiritus (2024) is that the marginal benefit from the public good policy differs across types due to different preferences for risk. Hence, the optimal policy requires not just balancing the volatility of the public good with that of private consumption, as in the cases of Schindler (2008) and Boadway and Spiritus (2024), but also requires to balance the different tastes for the public good.

The second strand concerns optimal taxation under heterogeneous returns to capital. Much of this literature generates heterogeneous returns through heterogeneous investment ability or scale effects (Boadway and Spiritus, 2024; Gahvari and Micheletto, 2016; Gerritsen et al., 2025; Kristjánsson, 2024; Saez and Stantcheva, 2018) as drivers of heterogeneous returns. This paper, in line with new empirical evidence (Bach et al., 2020), considers different preferences for risk as an alternative driver of heterogeneous returns. This creates a connection between savings and returns heterogeneity, and preference heterogeneity, as returns stem endogenously from preferences for risk through type-specific portfolio choices. This paper derives optimal tax conditions for labour and capital income, taking into account their spillover effects on other tax bases, and highlighting the key mechanisms at play under risk aversion heterogeneity. Most of the above contributions argue in favour of taxing or subsidising the riskless (normal) return on grounds of equity, as the taxation of safe capital income complements the redistributive role of earnings taxation.[7] When return heterogeneity stems from risk preferences, the direction of redistribution is governed by the planner's chosen normative criteria. At the joint optimum, the main role of the riskless-return tax is to redistribute across risk-aversion types within income levels, exploiting the fact that, conditional on income, more risk-averse agents save less. Whether these within-income transfers should favour the more or the less risk-averse – and hence whether the optimal rate is a tax or a subsidy – ultimately depends on the welfare weights that are used to compare utilities across preference types (Lockwood and Weinzierl, 2015; Saez, 2002). Moreover, I characterise, as an extension, the nonlinear taxation of expected capital income, showing that this instrument can screen on risk aversion through an additional margin – portfolio composition – since its elasticity is a composite of the savings and portfolio-share margins. This comes at the cost of breaking the Domar–Musgrave neutrality result that makes the linear excess-return tax distortion-free, introducing a first-order distortion to the savings margin.

2 Public Good Provision in a Risky Environment

One fundamental result of the theory of public goods is that, in general, pure public goods will be under-supplied by the market when provided on the basis of individual voluntary contributions due to the free-riding problem. This result remains true in a risky environment when agents possess different risk preferences. In this section, I argue that there is also another type of inefficiency, which pertains the distribution of risk between public and private consumption across states of nature.

Consider an economy with two periods and n agents indexed by $j = 1, \dots, n$, differing only in their coefficient of relative risk aversion $\theta_j > 0$. In period-0, agent j has endowment z and chooses first-period consumption c_0^j, savings $a_j = z - c_0^j$, and a contribution share $\delta_j \in [0, 1]$ of their period-1 wealth to allocate to the public good. In period 1, savings earn a risky gross portfolio return $\tilde{R}_p^j$, with $\mathbb{E}[\tilde{R}_p^a] > \mathbb{E}[\tilde{R}_p^b]$ and

[7] For instance, Boadway and Spiritus, 2024 shows that agents with lower investment ability obtain lower returns, save less, and carry higher welfare weights, making a subsidy on the normal return optimal.

$Var(\tilde{R}_p^a) > Var(\tilde{R}_p^b)$ when $\theta_a < \theta_b$. Hence, we have:

$$\tilde{c}_1^j = (1 - \delta_j)\, a_j\, \tilde{R}_p^j, \tag{1}$$

$$\tilde{P} = \sum_{k=1}^{n} \delta_k\, a_k\, \tilde{R}_p^k. \tag{2}$$

The public good $\tilde{P}$ is risky because it is funded from period-1 wealth, which depends on the realisation of $\tilde{R}_p^k$. The stochasticity of $\tilde{P}$ arises endogenously from the same aggregate shock that drives private consumption risk. The individual maximises lifetime expected utility given by (3). Without loss of generality, I impose no discounting of future consumption. Moreover, individuals evaluate private and public consumption with the same sub-utility function $u(\cdot)$, meaning that no specific difference in taste between private and public goods is modelled.

$$U^j = u_j(c_0^j) + \mathbb{E}\Big[u_j(c_1^j)\Big] + \mathbb{E}\Big[u_j(\tilde{P})\Big]\,. \tag{3}$$

Taking all other agents' contributions as given, agent j chooses the private contribution rule δ_j to satisfy

$$\mathbb{E}\Big[u_j'(\tilde{P})\,\tilde{R}_p^j\Big] \;=\; \mathbb{E}\Big[u_j'(\tilde{c}_1^j)\,\tilde{R}_p^j\Big]\,. \tag{4}$$

This equates the risk-adjusted expected marginal utility of public consumption to that of private consumption, where the weighting by $\tilde{R}_p^j$ reflects the opportunity cost of the marginal unit contributed. Define the risk-adjusted marginal rate of substitution: $MRS_j^r \;\equiv\; \frac{\mathbb{E}\big[u_j'(\tilde{P})\,\tilde{R}_p^j\big]}{\mathbb{E}\big[u_j'(\tilde{c}_1^j)\,\tilde{R}_p^j\big]}$. Condition (4) states $MRS_j^r = 1$ for each agent-type at the Nash equilibrium.

Then, we consider the solution of a social planner, who chooses the contribution shares δ_j to maximise a weighted sum of utilities $SW = \sum_j \omega_j U^j$, with individual welfare weights ω_j. The planner's FOC for δ_j is:

$$\omega_j\; \mathbb{E}\Big[u_j'(\tilde{c}_1^j)\,\tilde{R}_p^j\Big] \;=\; \sum_k \omega_k\; \mathbb{E}\Big[u_k'(\tilde{P})\,\tilde{R}_p^k\Big]\,. \tag{5}$$

The planner equates the weighted marginal cost to agent j to the social risk-adjusted marginal benefit, related to all agents k. Condition (5) characterises the (ex-ante) first best distribution of the public good, and therefore the (ex-ante) first-best distribution of risk between public and private consumption. Dividing (5) by the marginal cost to the contributor yields the risk-adjusted Samuelson rule for agent j's contribution:

$$\sum_k \frac{\omega_k\, \mathbb{E}\Big[u_k'(\tilde{P})\, \tilde{R}_p^k\Big]}{\omega_j\, \mathbb{E}\Big[u_j'(\tilde{c}_1^j)\, \tilde{R}_p^j\Big]} \;=\; 1 \;\text{ for every } j. \tag{6}$$

Proposition 1. *Under private provision with aggregate risk, the Nash equilibrium is inefficient in two distinct ways:*

(i) *Underprovision. The expected level of the public good under Nash equilibrium is lower than under first-best:* $\mathbb{E}[\tilde{P}^{NE}] < \mathbb{E}[\tilde{P}^{FB}]$*,*

(ii) *Suboptimal risk distribution. When agents have heterogeneous risk aversion* and *there is aggregate risk, the probability distribution of* $\tilde{P}$ *over states of nature is (ex-ante) suboptimal, with respect to first-best, except in degenerate cases.*

Point (i) in Proposition 1 is the standard underprovision result that applies also in the case of heterogeneous risk preferences. Proof of (i), that is similar to the standard case, is provided in the appendix A.1.

Proof of Part (ii). I show that there exists a feasible reallocation of contributions across agents that keeps $\mathbb{E}[\tilde{P}]$ unchanged but raises social welfare $SW = \sum_j \omega_j U^j$. The perturbation shows the Nash equilibrium fails to maximise SW, i.e. its distribution of $\tilde{P}$ is not first-best optimal.

Step 1. Let's consider the welfare gradient with respect to the contribution rate δ_j for an agent-type j at the Nash equilibrium, holding all other δ_k and all a_k fixed:

$$\frac{\partial SW}{\partial \delta_j} = a_j^* \left[-\omega_j \mathbb{E}\left[u_j'\left(\tilde{c}_1^j\right)\tilde{R}_p^j\right] + \sum_k \omega_k \mathbb{E}\left[u_k'(\tilde{P})\tilde{R}_p^j\right]\right].$$

At the Nash equilibrium, (4) gives $\mathbb{E}\left[u_j'(\tilde{P})\tilde{R}_p^j\right] = \mathbb{E}\left[u_j'\left(\tilde{c}_1^j\right)\tilde{R}_p^j\right]$, so the $k = j$ terms cancel and:

$$\left.\frac{\partial SW}{\partial \delta_j}\right|_{\text{NE}} = a_j^* \sum_{k \neq j} \omega_k \mathbb{E}\left[u_k'(\tilde{P})\tilde{R}_p^j\right] > 0.$$

The marginal effect of δ_j on $\mathbb{E}[\tilde{P}]$ is $\partial \mathbb{E}[\tilde{P}]/\partial \delta_j = a_j^* \mathbb{E}\left[\tilde{R}_p^j\right]$. Then, the normalised welfare gradient - the welfare gain per unit increase in $\mathbb{E}[\tilde{P}]$ from raising agent j 's contribution - will be given by:

$$\Gamma_j \equiv \frac{\partial SW/ \left.\partial \delta_j\right|_{\text{NE}}}{a_j^* \mathbb{E}\left[\tilde{R}_p^j\right]} = \frac{\sum_{k \neq j} \omega_k \mathbb{E}\left[u_k'(\tilde{P})\tilde{R}_p^j\right]}{\mathbb{E}\left[\tilde{R}_p^j\right]}.$$

Step 2: suppose we change contribution for agent-types i and j such that the expected level of the public good $\mathbb{E}[\tilde{P}]$ remains constant, but the risk-profile across states of nature is different. Holding $\mathbb{E}[\tilde{P}]$ constant means that $d\,\mathbb{E}[\tilde{P}] = dg_i + dg_j = 0$ and $dg_j = -dg_i$, with $dg_i = a_i\,\mathbb{E}\left[\tilde{R}_p^i\right] d\delta_i$ being the marginal change in the value of contribution due to the marginal change of the contribution share δ. The resulting change in welfare is:

$$dSW = \frac{\partial SW}{\partial g_i} dg_i + \frac{\partial SW}{\partial g_j} dg_j \overset{dg_j = -dg_i}{=} \left(\frac{\partial SW}{\partial g_i} - \frac{\partial SW}{\partial g_j}\right) dg_i$$

Dividing by dg_i, you get the constrained rate of change:

$$\left.\frac{dSW}{dg_i}\right|_{dg_j = -dg_i} = \frac{\partial SW}{\partial g_i} - \frac{\partial SW}{\partial g_j} = \frac{\partial SW/\partial \delta_i}{a_i\,\mathbb{E}\left[\tilde{R}_p^i\right]} - \frac{\partial SW/\partial \delta_j}{a_j\,\mathbb{E}\left[\tilde{R}_p^j\right]}$$

that is the difference between the welfare gains from raising the contributions of agent-type i and j respectively, normalised by the corresponding increases in $\mathbb{E}[\tilde{P}]$. We can rewrite the last expression using the welfare gradients directly:

$$\left.\frac{dSW}{dg_i}\right|_{dg_j = -dg_i} = \Gamma_i - \Gamma_j. \tag{7}$$

Due to heterogeneous preferences, consumption and portfolio decisions will be different across agents, and so will the welfare gradients. Hence, except for degenerate cases, expression (7) is likely to be non-zero meaning it is possible to reshuffle the contribution rates, δ_i and δ_j, to increase social welfare, while keeping fixed the expected level of the public good.

□

3 Public Good Provision with Linear Taxes

In this section, we explicitly introduce taxes as a way to finance the provision of the risky public good. In this section, we consider linear taxes on savings that are set to maximise the weighted sum of utilities $SW = \sum_j \omega_j U^j$, with individual welfare weights ω_j and utility U_j given by (3). Two benchmark cases are presented: 1) Observable types: the government can levy type-specific proportional taxes that depend on individual risk aversion (section 3.1); 2) Non-observable types: the government sets a unique proportional tax rate for all types of agent (section 3.2).

3.1 Observable Types

When types are observable, the government can choose type-specific proportional taxes $\tau(\theta_j)$ that perfectly target agents with different risk preferences. Consider the utility function given by expression (3). Private and public consumption will be given by:

$$\tilde{c}_1^j = (1 - \tau(\theta_j))\, a_j\, \tilde{R}_p^j, \tag{8}$$

$$\tilde{P} = \sum_{k=1}^{n} \tau_k\, a_k\, \tilde{R}_p^k, \tag{9}$$

where the public good is stochastic, as it is affected by aggregate shocks to the portfolio returns $\tilde{R}_p^k$. Then, the optimal $\tau(\theta_j)$ will equate the marginal private costs of agent j from contributing to the public good, expressed in terms of private consumption, to the social benefit of a marginal unit, taking into account the distortionary costs of taxation. The resulting optimality condition is:

$$p(\theta_j) \underbrace{\left[1 - \frac{\tau(\theta_j)}{1 - \tau(\theta_j)} \varepsilon_{a,\tau}(\theta_j) - \tau(\theta_j) \cdot \frac{\eta_{a,\tau}(\theta_j)}{a_j(\theta_j)}\right]}_{\Psi(\tau_j, \theta_j)} = 1 \tag{10}$$

where $p(\theta_j) = \frac{\mathbb{E}\left[\sum_k \omega_k u_k'(\tilde{P})\tilde{R}_p^j\right]}{\mathbb{E}\left[\omega_j u_j'(\tilde{c}_1^j)\tilde{R}_p\right]}$ is the ratio between the social welfare gain from an additional unit of public good for all types through a higher tax on agent-type j, and the private welfare cost for that agent j. The compensated behavioural effect is captured by the elasticity of savings to the net-of-tax rate $\varepsilon_{a,\tau}(\theta_j) = \frac{\partial a(\theta)}{\partial(1-\tau)} \cdot \frac{(1-\tau)}{a(\theta)}\Big|_{\overline{U}}$, whereas $\eta_{a,\tau}(\theta_j)$ is the income effect. Overall, the term $\Psi(\tau_j, \theta_j)$ captures the mechanical effect of raising the tax on one agent-type, net of behavioural responses. Lemma 1 provides the corresponding optimal tax formula. The derivation is provided in appendix A.2.

Lemma 1. *When different risk-aversion types of agents are targeted individually, the optimal linear tax on savings is given by*

$$\frac{\tau(\theta_j)}{1-\tau(\theta_j)} = \frac{1}{a(\theta_j)\varepsilon_{a,\tau}(\theta_j)} \cdot \left[\frac{a(\theta_j)\cdot(p(\theta_j)-1)}{p(\theta_j)} - \tau(\theta_j)\cdot\eta_{a,\tau}(\theta_j)\right] \tag{11}$$

For each agent-type, the tax is inversely related to the compensated elasticity of savings $\varepsilon_{a,\tau}(\theta_j)$ and will be positive provided that $p(\theta_j)$ is sufficiently larger than 1, and/or income effect is small. If the compensated response of an agent-type with low risk aversion is larger than the one of a high risk aversion agent, then they would face a lower tax rates than high risk aversion individuals.

3.2 Unobservable Types

If the government cannot screen agents with different risk preferences, and therefore cannot set (risk-aversion) type specific taxes, a uniform proportional tax τ^u applies to all agents. The optimality condition (12) equates marginal social costs, expressed in terms of private consumption, to the social benefits, net of distortionary costs.

$$\mathbb{E}\left[\sum_k \omega_k u_k'(\tilde{P}) \sum_j \tilde{R}_p^j a_j \Psi\left(\tau^u, \theta_j\right)\right] - \mathbb{E}\sum_k \left[\omega_k u_k'(\tilde{c}_1^k)\tilde{R}_p^k a_k\right] = 0 \tag{12}$$

where $\Psi\left(\tau^u, \theta_k\right) = \left(1 - \frac{\tau^u}{1-\tau^u}\varepsilon_{a,\tau}^k(\theta_k) - \tau^u \cdot \frac{\eta_{a,\tau}}{a_k}\left(\theta_k\right)\right)$. Proposition 2 states that under uniform taxation of agent-types, we have lower provision of the public good and less efficient risk allocation, compared to the targeted system given by (11).

Proposition 2. *The risk allocation between private and public consumption under the unique proportional tax condition (12) is welfare-inferior relative to the case of the type-specific tax rule (11). Moreover, the provision of the public good will be lower in expectation.*

Proof. Part 1: Inefficient risk allocation. Suppose we deviate from the unique tax rule and marginally change the tax for two agent-types such that the expected level of the public good $\mathbb{E}[\tilde{P}]$ remains constant, but the risk-profile across states of nature is different. Holding $\mathbb{E}[\tilde{P}]$ constant means that $d\,\mathbb{E}[\tilde{P}] = a_j\,\mathbb{E}\left[\tilde{R}_p^j\right]\Psi(\tau^u,\theta_j)d\tau_j + a_i\,\mathbb{E}\left[\tilde{R}_p^i\right]\Psi(\tau^u,\theta_i)d\tau_i = 0$, such that $d\tau_i = -\frac{a_j\,\mathbb{E}\left[\tilde{R}_p^j\right]}{a_i\,\mathbb{E}\left[\tilde{R}_p^i\right]}\cdot\frac{\Psi(\tau^u,\theta_j)}{\Psi(\tau^u,\theta_i)}d\tau_j$. We can rewrite the change in welfare $dSW = \frac{\partial SW}{\partial\tau_j}d\tau_j + \frac{\partial SW}{\partial\tau_i}d\tau_i$ as:

$$\left.\frac{dSW}{d\tau_j}\right| = \underbrace{\mathbb{E}\left[\omega_j u_j'(\tilde{c}_1^j)\tilde{R}_p^j\right]a_j\mu_j}_{\frac{\partial SW}{\partial\tau_j}} - \underbrace{\mathbb{E}\left[\omega_i u_i'(\tilde{c}_1^i)\tilde{R}_p^i\right]a_i\mu_i}_{\frac{\partial SW}{\partial\tau_i}}\cdot\frac{a_j\,\mathbb{E}\left[\tilde{R}_p^j\right]}{a_i\,\mathbb{E}\left[\tilde{R}_p^i\right]}\frac{\Psi(\tau^u,\theta_j)}{\Psi(\tau^u,\theta_i)}$$

where $\mu = [p(\theta)\Psi(\tau^u,\theta) - 1]$. The term $p(\theta)\Psi(\tau^u,\theta)$ is a measure of the net social return of funding the public good by raising the tax on one agent-type. When the net social return is above 1 ($\mu_j > 0$), agent-type j is undertaxed on the basis of the uniform tax τ^u relative to what condition (11) would require. Hence, it is possible to raise welfare by deviating from the unique tax rule τ^u, namely choosing $\tau_j > \tau^u$ for agent-types for which we have $\mu_j > 0$, and choosing $\tau_i < \tau^u$ for those agents for which $\mu_i < 0$. This alternative policy impacts the state-contingent risk distribution in the economy without changing the expected value of the public good. It follows that the unique tax rule characterises a suboptimal equilibrium relative to condition (11).

Part 2: Underprovision. To prove this point, first notice that aggregating condition (10) across types gives the following formula:

$$\mathbb{E}\left[\sum_k \omega_k u_k'(\tilde{P})\sum_j \tilde{R}_p^j a_j \Psi\left(\tau(\theta_j),\theta_j\right)\right] - \mathbb{E}\left[\sum_j \omega_j u_j'\left(\tilde{c}_1^j\right)\tilde{R}_p^j a_j\right] = 0$$

where $\Psi\left(\tau(\theta_j),\theta_j\right) = \left(1 - \frac{\tau(\theta_j)}{1-\tau(\theta_j)}\varepsilon_{a,\tau}\left(\theta_j\right) - \tau\left(\theta_j\right)\frac{\eta_{a,\tau}}{a_j}\left(\theta_j\right)\right)$. This condition has the same structure as condition (12). However, tax rates $\tau(\theta_j)$ are optimised across different types of agent in condition (10) accounting for the type-specific behavioural responses to taxes. The uniform tax τ^u is a restricted version of the targeted system $\tau(\theta_j)$, which can raise any given level of expected revenue at a lower overall private welfare costs compared to τ^u. Indeed, under the targeted system $\tau(\theta_j)$, it is possible to shift the tax burden towards agents with lower elasticities or social welfare weights, i.e. the marginal cost of public funds is lower under the targeted system. If the expected social benefit of the public good is decreasing, then under the targeted system public provision will be higher in expectation as, compared to the unique tax case τ^u, a lower marginal benefit of the public good will equate a lower marginal cost of public funds. □

4 Heterogeneous Productivity and Risk Aversion

In this section, I consider a more complex environment in which each type of agent is characterised by their labour productivity and risk aversion. The government taxes labour earnings with a non-linear schedule and capital income linearly, distinguishing the types of return from investment, i.e. riskless (normal) return versus risky excess return. Alternatively, non-linear taxation of expected capital income is considered. Agents with different risk preferences choose different amount of hours worked, conditional on their productivity (wage).

4.1 Set-up

The economy is populated by a continuum of agents in the type space $\Omega = \{w, \theta\}$ who differ along two dimensions: pre-tax income z due to unobserved ability w, and risk aversion θ. Agents choose their labour supply l, first period consumption c_0, and the portfolio composition of their savings. Agents invest their savings $a(z, \theta)$ choosing the portfolio share of risky assets $s(\theta)$ with the risky return $\tilde{r}_e$ being normally distributed: $\tilde{r}_e \sim \mathcal{N}(\overline{r}_e - \sigma_r^2/2, \sigma_r^2)$. The resulting portfolio return $\tilde{r}_p$ will also be normally distributed: $\tilde{r}_p \sim \mathcal{N}(\overline{r}_p - \sigma_p^2/2, \sigma_p^2)$. The (risky) excess return is defined as the difference between the risky return and the riskless return: $\tilde{r}^{exc} := \tilde{r}^e - r$. Without loss of generality, I impose no time discounting of future consumption.

Labour income is taxed non linearly under the tax schedule $T(z)$. Each component of capital income is taxed linearly: τ_k is the tax on the excess return, τ_r is the tax rate on the safe return. As for behavioural responses to taxes, the compensated elasticities of labour earnings z and savings a with respect to net-of-marginal-tax rate on labour earnings, $\varepsilon_{z,l}$, $\varepsilon_{a,l}$, and with respect to net-of-tax rate on safe capital income, $\varepsilon_{z,r}$, $\varepsilon_{a,r}$, are defined as follows:

$$\varepsilon_{z,l}(w,\theta) \equiv \frac{\partial z(w,\theta)}{\partial(1-T'(w))} \cdot \frac{1-T'(w)}{z(w,\theta)}\bigg|_{\overline{U}}; \qquad \varepsilon_{z,r}(w,\theta) \equiv \frac{\partial z(w,\theta)}{\partial(1-\tau_r)r} \cdot \frac{(1-\tau_r)r}{z(w,\theta)}\bigg|_{\overline{U}};$$

$$\varepsilon_{a,l}(w,\theta) \equiv \frac{\partial a(w,\theta)}{\partial(1-T'(w))} \cdot \frac{1-T'(w)}{a(w,\theta)}\bigg|_{\overline{U}}; \qquad \varepsilon_{a,r}(w,\theta) \equiv \frac{\partial a(w,\theta)}{\partial(1-\tau_r)r} \cdot \frac{(1-\tau_r)r}{a(w,\theta)}\bigg|_{\overline{U}}.$$

Finally, $\eta_l(w,\theta)$ and $\eta_a(w,\theta)$ are the income effects on labour earnings and riskless savings respectively.

4.2 Agent's Problem

Agent's utility is separable over time, and separable between private and public consumption. Agents choose $\{c_0(w,\theta), s(\theta), l\}$ to maximise

$$U(w,\theta) = u(c_0(w,\theta)) + \mathbb{E}\left[u(\tilde{c}_1(w,\theta)) + u(\tilde{P})\right] - v(l) \tag{13}$$

subject to $\tilde{c}_1(w,\theta) = \tilde{R}_p(\theta)a(w,\theta)$, where $a(w,\theta) = z(w,\theta) - c_0(w,\theta) - T(z)$ is the value of assets, given labour earnings $z = wl$; $\tilde{R}_p(\theta)$ is the gross portfolio return defined as

$$\tilde{R}_p(\theta) = \left[1 + r\left(1-\tau_r\right) + s_j\left(\tilde{r}^{exc}\right)\left(1-\tau_k\right)\right],$$

and the gross riskless return is identified by R. As each agent is infinitesimal compared to the size of the economy, the effects of individual choices on the level of the public good are not taken into account.

$$c_0^*: \qquad u'(c_0(w,\theta)) = \mathbb{E}\left[u'(\tilde{c}_1(w,\theta))\tilde{R}_p\right] \tag{14}$$

$$l^*: \qquad 0 = \mathbb{E}\left[u'(\tilde{c}_1(w,\theta))\tilde{R}_p\right][1-T'(z)]w - v'(l) \tag{15}$$

$$s^*: \qquad 0 = \mathbb{E}\left[u'(\tilde{c}_1(w,\theta))\tilde{r}^{exc}\right] \tag{16}$$

Conditions (14)-(15) are the standard first order conditions (FOCs) for first-period consumption c_0, and labour supply l. A close form solution for first-period consumption can be obtained (see appendix B.2). On the basis of this, the following lemma can be stated with regard to agents' saving behaviour.

Lemma 2. *For relative risk aversion parameters $\theta \in (0,2)$, savings decrease with risk aversion.*

Proof available in appendix B.3.

Then, expression (16) is the FOC for the share of risky assets s, determining the portfolio allocation with risky and riskless assets. After applying the covariance identity to condition (16), we can derive the expression for the risk premium with respect to the expected excess return $\mathbb{E}[R^e - R]$.[8]

$$\mathbb{E}[R^e - R] = -\frac{\text{cov}\left[u'(\tilde{c}_1(w,\theta)), \tilde{r}^{exc}\right]}{\mathbb{E}\left[u'(\tilde{c}_1(w,\theta))\right]}. \tag{17}$$

For each agent with productivity w and relative risk aversion parameter θ, expression (17) maps the covariance term to the expected marginal utility of second-period consumption $\tilde{c}_1$. Under constant relative risk aversion utility, a close form solution for the share of risky assets can also be derived for any parameter $\theta > 0$ (with $\theta \neq 1$):[9]

$$s^*(\theta) = \frac{\overline{r}^e - r}{\theta\sigma_r^2} \cdot \frac{(R - r\tau_r)}{R(1-\tau_k)} \tag{18}$$

While the first term relates to the original formula by Merton (1969) and Samuelson

[8] R^e and R are the gross risky return and gross riskless return respectively.

[9] The derivation is provided in the appendix B.1.

(1969),[10] the second term captures the effect of the different taxes on capital income. In particular, increasing the tax on safe capital income reduces the optimal share of risky assets: $\frac{\partial s^*}{\partial \tau_r} < 0$. On the contrary, increasing the excess return tax, while keeping the loss offsets, increases $s^*(\theta)$: $\frac{\partial s^*}{\partial \tau_k} > 0$. Agents will adjust their portfolio shares to get the same post-tax expected portfolio return (Domar and Musgrave, 1944).

Individuals' optimal labour supply, savings and portfolio decisions map the primitive type space $\Omega = (w, \theta)$ into an induced joint distribution of observable earnings z and unobserved risk aversion θ, defined over $\Omega_z = (z, \theta)$. The government cannot observe an individual's risk preference, so its instruments are restricted to functions of observables: the schedule $T(z)$ conditions on earnings alone, and τ_r and τ_k are uniform across types. Risk aversion enters the problem only through the aggregation: the government is assumed to know the induced joint distribution $f(z, \theta)$, and evaluates tax reforms using averages taken over the conditional distribution $f(\theta|z)$ at each income level.

4.3 The Government

The government's objective is to maximise social welfare (SW) that is defined as an increasing and concave function Φ of agents' lifetime expected utilities U (as in (13)), given population density at income z and risk aversion θ, $f(z, \theta)$:

$$SW = \int_{\Omega_z} f(z, \theta)\, \Phi\left(U\left(z, \theta\right), \theta\right) d(z, \theta).$$

Notice that the function $\Phi\left(U\left(z, \theta\right), \theta\right)$ depends itself on the risk aversion parameter. These risk-aversion-specific transformations are necessary to accommodate different individual risk preferences as well as concerns for fairness (Grant et al., 2010).[11]

The government provides a public good $\tilde{P}$ that enters the utility function separately from private consumption. The public good is financed by taxes on labour earnings, through a non-linear tax schedule $T(z)$, and on capital income, through lin-

[10] Merton (1969) and Samuelson (1969) study the optimal portfolio choice of a consumer with CRRA utility with a multi-period horizon and multiple risky assets.

[11] The issue of choosing the appropriate functions for utility functions with different curvatures is widely debated in the welfare theory literature (Eden, 2020; Grant et al., 2010), but it is beyond the scope of this paper.

ear taxes on the risky excess return, τ_k, and on the safe return at rate τ_r. The public good is given by the following expression:

$$\tilde{P} = \int_{\Omega_z} f(z,\theta)T(z)d(z,\theta) + \int_{\Omega_z} f(z,\theta)\left(\tau_k s^*(\theta)\,\tilde{r}^{exc} + \tau_r r\right)a(z,\theta)d(z,\theta). \tag{19}$$

Tax rates are set by the government in the first period anticipating agents' optimal behaviour. After the state of the economy is realised, the government implements the public good policy and balances the budget, meaning that the budget is balanced for any state of the world. That is a stricter requirement than balancing the budget in expectation, that would instead imply transferring resources from good states to bad states of the world. As a result, the provision of the public good is stochastic and depends on the state of the economy in the second period, determining the value of revenues from the taxation of excess returns on savings. An additional unit of tax revenues used to finance the provision of public good $\tilde{P}$ in a given state of nature generates the welfare effect

$$\tilde{\alpha} = \frac{1}{\lambda}\int_{\Omega_z} \omega(z,\theta)\, u'(\tilde{P};z,\theta)\cdot f\,(z,\theta)\, d(z,\theta), \qquad \alpha \equiv \mathbb{E}[\tilde{\alpha}] \tag{20}$$

with $\omega(z,\theta) = \frac{\partial\Phi(U(z,\theta),\theta)}{\partial U(z,\theta)}$ and $u'(\tilde{P};z,\theta) = \partial U(z,\theta)/\partial\tilde{P}$. The random variable $\tilde{\alpha}$ is the state-contingent marginal value of public funds; its ex-ante expectation α is the social welfare gain per unit of tax revenues devoted to public good provision, normalized by the marginal value of public funds λ, the shadow price of the government budget constraint in units of social welfare.

Finally, following Saez (2001), I define the social marginal welfare weight

$$g(z,\theta) \equiv \frac{\omega\,(z,\theta)\cdot U_c(z,\theta)}{\lambda} \geq 0, \tag{21}$$

where $U_c(z,\theta) = \mathbb{E}\left[u'(\tilde{c}_1)\tilde{R}_p\right]$ is the return-weighted expected marginal utility of second-period consumption, and $\omega\,(z,\theta)$ are the type-specific weights capturing both the fairness motive and the cardinal normalisation of utility across agents with different risk preferences.

4.4 Optimal Non-linear Taxation of Labour Income

The labour income tax schedule is modeled with an intercept and non-linear component: $T(z) = \tau_0 + \int_0^z T'(z')dz'$. The optimality conditions are obtained using a perturbation approach by considering small welfare-neutral variations to the tax rates (Saez, 2001). Proposition 3 presents the optimality conditions. The derivations are provided in Appendix A.3.

Proposition 3. *The optimal labour income tax schedule is described by the following two conditions:*

- *Marginal tax rates* $T'(z)$*:*

$$\frac{T'(z)}{1-T'(z)} = \frac{1}{z\,\overline{\varepsilon}_{z,l}}\left[\frac{(1-H(z))(\alpha-\overline{G}(z))}{\alpha\, h(z)} - \frac{\overline{\varepsilon}^{a}_{a,l}\cdot\tau_r r + \overline{\varepsilon}^{a}_{k,l}\cdot\tau_k\ \mathbb{E}[\tilde{\alpha}\tilde{r}^{\text{exc}}]/\alpha}{1-T'(z)}\right.$$
$$\left. + \frac{1}{h(z)}\int_z^{\infty}\left(\overline{\eta}_l(z')\,T'(z') - \overline{\eta}^{a}_l(z')\,\tau_r r - \frac{\mathbb{E}[\tilde{\alpha}\tilde{r}^{\text{exc}}]}{\alpha}\cdot\overline{\eta}^{k}_l(z')\tau_k\right)h(z')dz'\right]. \quad (22)$$

- *Intercept* τ_0*:*

$$\alpha = \int_{\Omega_z} g(z,\theta)\cdot f(z,\theta)d(z,\theta). \quad (23)$$

The optimal marginal tax rate on labour income is set by condition (22). It is obtained by considering a small reform in which the marginal tax rate is increased by $d\tau_l$ in a small earnings band $(z, z+\Delta z)$, with no change elsewhere in the earnings distribution. Expression (22) has a similar structure to Saez (2001). The elasticity parameter $\overline{\varepsilon}_{z,l} = \int \varepsilon_{z,l}(z,\theta)\cdot f(\theta \mid z)d\theta$, averaging elasticity across risk-aversion types at income level z, and the hazard rate $h(z)/(1-H(z)))$, capture the efficiency channel, that drives marginal tax rates down. The term $(\alpha - \overline{G}(z))$ captures the net social gain from raising marginal taxes on those above z to finance the public good: α is the normalised social welfare gain from an additional unit of public good, and $\overline{G}(z) = \frac{\int_z^{\infty}\overline{g}(z')\,h(z')dz'}{1-H(z)}$ is the average social marginal welfare weight above income level z, with $\overline{g}(z') = \int g(z',\theta)\cdot f(\theta|z')\,d\theta$ being the average social marginal welfare weight at income level z' across agents with different risk aversion parameters.

A concave social welfare function generally implies the social welfare weights decline along the income distribution. However, conditional on income, agents with different risk aversion will also have different expected marginal utilities and social marginal welfare weights. Therefore, if income is somewhat correlated with risk aversion, the shape of the labour income tax schedule depends on the joint distribution of income and risk preferences in the population as well as the set of weights $\omega(z,\theta)$. Two redistributive objectives are at play here. On the one hand, the government would like to redistribute from high income agents to low income agents for equity reasons. On the other hand, conditional on income, the agents with lower risk aversion have a higher expected marginal utility of consumption than agents with higher risk aversion. However, since the net social benefit of any policy is evaluated through the social marginal welfare weights (21), the direction of redistribution across risk aversion types will ultimately depend on the set of weights $\omega(z,\theta)$ used to evaluate social welfare. For instance, if risk aversion is inversely correlated with income, and the weights increase with risk aversion conditional on income, then the planner's desire to redistribute toward low-income individuals aligns with the desire to redistribute toward more risk-averse individuals, both pushing toward a more progressive labour tax schedule. But if the welfare weights decline with risk aversion, a tension arises between these two objectives, resulting in a less progressive tax schedule, all else being equal.

Another point worth discussing relates to spillovers effects of labour income tax changes to other tax bases. Raising the marginal tax rate on labour income has fiscal externalities on capital income tax revenues. Compensated effects on capital income are captured by the elasticity term $\bar{\varepsilon}^a_{a,l} = \int \varepsilon_{a,l}(z,\theta)\cdot a(z,\theta)f(\theta|z)d\theta$ that is the mean compensated elasticity of savings with respect to the net-of-tax savings, averaged over risk aversion types at income z, weighted by savings, and by the corresponding savings-weighted elasticity term for risky excess capital income $\bar{\varepsilon}^a_{k,l} = \int \varepsilon_{k,l}(z,\theta)\cdot s(\theta)a(z,\theta)f(\theta|z)d\theta$. Given that $T(z)$ is set ex-ante, the fiscal externality term involving the risky excess return is evaluated on the basis of the expectation of the product between the state-contingent marginal value of public funds and the excess return, $\mathbb{E}[\tilde{\alpha}\tilde{r}^{exc}]$. In section 4.5.1, I show that this fiscal externality vanishes exactly once the excess-return tax is set optimally, such that condition (22) is well defined at the optimum. Then, the last term of condition (22) captures the income effects.[12]

Finally, as for the intercept τ_0, condition (23) equates the social value of extracting one more unit of revenue from the population through the lump-sum component τ_0 to the marginal social welfare of the public good α.

4.5 Optimal Linear Capital Income Taxation

This section explores the linear taxation of the two components of capital income: excess risky return, and normal (riskless) return. First, section 4.5.1 focuses on the taxation of the risky excess return. Second, section 4.5.2 focuses on the linear taxation of the normal (riskless) return. This requires the government to be able to distinguish the different sources of capital income, which in turn requires observing initial and capitalised savings, and the market returns.

4.5.1 Taxation of Excess Return

The optimal tax rate on excess returns τ_k is set by condition (24). The revenues collected from taxing the excess return are risky as the realisation of the excess return depends on the state of the economy. Hence, the public good will have a certain probability distribution which entails a certain allocation of risk between private and public consumption. Condition (24) determines the variance of the public good policy and the allocation of risk between private and public consumption that maximises social welfare.[13] In other terms, the government is choosing the optimal risky share of the budget (tax revenues), given agents' preferences and social welfare weights.

[12] Income effects are captured by the following terms averaging income effects across risk-aversion types at a given income level: (i) $\overline{\eta}_l(z')T'(z') = T'(z')\int_\theta \eta_l(z',\theta)f(\theta|z')\,d\theta$ for labour income; (ii) $\overline{\eta}_l^a(z')\tau_r r = \tau_r r\int_\theta \eta_a(z',\theta)\cdot a(z',\theta)\cdot f(\theta|z')\,d\theta$ for riskless capital income; (iii) $\overline{\eta}_l^k(z')\,\tau_k\tilde{r}^{\mathrm{exc}} = \tau_k\tilde{r}^{\mathrm{exc}}\int \eta_k(z',\theta)\cdot s(\theta)a(z',\theta)\cdot f(\theta \mid z')\,d\theta$ for the risky asset holdings.

[13] When a bad (good) state of economy realises in the second period, losses (gains) due to negative (positive) excess returns will be spread over private and public consumption, so that the utility loss (gain) is minimised (maximised).

Proposition 4. *The optimal excess return tax is set by the following condition:*

$$\mathbb{E}\left[\tilde{r}^{exc}\int_{\Omega_z}\omega\left(z,\theta\right)u'(\tilde{P};z,\theta)f(z,\theta)\,d(z,\theta)\right]=0. \tag{24}$$

The key novelty of this setting with respect to the literature, specifically the work by Schindler (2008) and Boadway and Spiritus (2024), is that the "benefits" of the public good differ among agents because of heterogeneous attitudes to risk. Since the public good is itself risky, the welfare gain from increasing the tax rate τ_k varies across different types of agent. Hence, the government aims to balance the different "tastes" for the public good, providing insurance against aggregate risk. As individuals make portfolio choices on the basis of their risk preferences according to condition (17), similarly the government decides the distribution of tax revenues financing the public good over states of nature on the basis of (24). In doing so, the government takes into account the agents' willingness to shift risk to the societal level. Condition (25) in corollary 1 reformulates (24) and better represents this concept. Private and public consumption volatility are represented by the covariance between marginal utility of private and public consumption respectively with the risky excess return and jointly govern the individual willingness to pay for the public good with an extra euro of risky capital income.

Corollary 1. *The optimal excess return tax τ_k^* equalizes aggregate public consumption volatility with aggregate private consumption volatility. The optimal tax rate is positive but strictly lower than 100%.*

$$\frac{\int_{\Omega_z}\omega\left(z,\theta\right)Cov\left(\tilde{r}^{exc},u'(\tilde{P};z,\theta)\right)f(z,\theta)\,d(z,\theta)}{\int_{\Omega_z}\omega\left(z,\theta\right)Cov\left(\tilde{r}^{exc},u'(\tilde{c}_1;\theta)\right)f(z,\theta)\,d(z,\theta)}=1 \tag{25}$$

Proof. See appendix A.4.

Corollary 1 says that τ_k^* is chosen on the basis of a weighted average of private consumption volatility, i.e. $\int_{\Omega_z}\omega\left(z,\theta\right)\mathrm{Cov}\left(\tilde{r}^{exc},u'(\tilde{c}_1;\theta)\right)f(z,\theta)\,d(z,\theta)$. This is because the government cannot target individual risk-aversion types. This creates an inefficiency: an agent with relatively (more) risky private consumption c_1, compared to other agents, would be better-off by shifting more risk to the public good. For that to be the case, a higher tax rate τ_k should be implemented, so that private consumption volatility is traded with public consumption volatility.

Setting the optimal excess return tax also has an effect on the labour earnings marginal tax condition (22).

Corollary 2. *When the excess return is set optimally, the risky-capital income fiscal externality in labour earnings marginal tax condition (22) drops out. As a result, the labour tax formula (22) is well defined at the joint optimum.*

Proof. Available in the appendix A.7.

Finally, I investigate whether the taxation of excess returns can have any redistributive role across types of agent. This is clarified by corollary 3.

Corollary 3. *The optimal taxation of the risky excess returns does not serve a redistributive role across different types of agent, either across different levels of productivity or different levels of risk aversion.*

Proof. Available in the appendix A.8.

Hence, the role of the risky excess return is solely to set the variability of tax revenues and the public good on the aggregate level. This extends the finding by Boadway and Spiritus (2024) that such tax cannot enhance redistribution on equity grounds when agents differ by their labour and investment productivity.

4.5.2 Taxation of Normal Return

Once the excess return is taxed optimally, introducing a tax or subsidy on the normal return (riskless capital income) is welfare enhancing when the following condition is satisfied:

$$\begin{aligned} \left.\frac{\partial SW}{\partial \tau_r}\right|_{\tau_r=0,\tau_k^*} &= -\operatorname{Cov}(g(z,\theta), a(z,\theta)) \\ &\quad + \alpha \int_{\Omega_z} T'(z)\left[\eta_l(z,\theta)\cdot a(z,\theta) - \varepsilon_{l,r}(z,\theta)\cdot z\right] f(z,\theta)\, d(z,\theta) \neq 0. \end{aligned} \tag{26}$$

Condition (26) is the social welfare effect of introducing τ_r. The first term refers to the mechanical and welfare effects of introducing the tax. The second term relates to the net revenue effect on labour taxes from a unit increase in τ_r: the positive income effect on labour earnings weighted by savings minus the negative cross substitution effect on labour earnings weighted by earnings. When condition (26) is satisfied, taxing (or subsidising) the normal (safe) return on savings complements the role of labour income taxation. Proposition 5 presents the optimal tax formula. The derivation is provided in the appendix A.5.

Proposition 5. *Suppose the tax on excess return is set optimally. Then, the optimal tax rate on the normal riskless return is given by:*

$$\frac{\tau_r}{1-\tau_r} = \frac{1}{\alpha\,\bar{\varepsilon}^a_{a,r}}\Bigg[-\operatorname{Cov}(g(z,\theta),a(z,\theta)) - \frac{\alpha}{r}\frac{\int_{\Omega_z}\varepsilon_{l,r}(z,\theta)\cdot z\,T'(z)f(z,\theta)\,d(z,\theta)}{1-\tau_r} + \alpha\int_{\Omega_z}\left[\eta_l(z,\theta)T'(z)-\eta_a(z,\theta)\tau_r\right]a(z,\theta)f(z,\theta)\,d(z,\theta)\Bigg] \tag{27}$$

where $\bar{\varepsilon}^a_{a,r} = \int_{\Omega_z}\varepsilon_{a,r}\cdot a(z,\theta)f(z,\theta)d(z,\theta)$ is the savings-weighted aggregate elasticity of riskless savings with respect to the net-of-tax rate, capturing the efficiency costs of taxing the normal return due to the intertemporal distortion of resource allocation. The redistributive concerns along the distribution of savings are represented by the term $\operatorname{Cov}(g(z,\theta),a(z,\theta))$. If the social marginal welfare weight declines (increases) along the distribution of savings, a positive (negative) tax rate on the normal return can be optimal, provided the sum of all the other terms on the right-hand side of (27) is positive (negative).[14]

To understand better the redistributive role of this tax, alongside the labour income tax schedule, it is useful to evaluate (27) at the joint optimum, when also $T(z)$ is optimally set. To do that, first consider that the marginal social welfare weight can vary both across income levels, and across risk aversion types within the same income levels. This can be seen by decomposing the covariance term using the

[14]The other terms of condition (27) represent the fiscal externality of labour income tax revenues, and the effects on tax revenues due to the income effects on labour income and savings. The fiscal externality on risky excess return is omitted because of Corollary 2.

law of total covariance:

$$\mathrm{Cov}\big(g(z,\theta),a(z,\theta)\big) \;=\; \underbrace{\int_z \mathrm{Cov}_\theta\big(g(z,\theta),a(z,\theta)\,\big|\,z\big)\,h(z)\,dz}_{\text{within-income}} \;+\; \underbrace{\mathrm{Cov}_z\big(\bar{g}(z),\bar{a}(z)\big)}_{\text{across-income}},$$

where $\bar{g}(z) \equiv \int_\theta g(z,\theta)f(\theta \mid z)\,d\theta$ and $\bar{a}(z) \equiv \int_\theta a(z,\theta)f(\theta \mid z)\,d\theta$ are the risk-aversion-averaged welfare weight and savings level at each income z. The across-income term relates to how the average welfare weight and the average savings level co-move with z. That is the main channel through which the optimal labour income tax schedule $T(z)$ works, that entails redistributing across income levels. Hence, when the labour income tax schedule $T(z)$ is set optimally, the distinctive role for optimal τ_r goes through the within-income component, handling redistribution across risk aversion levels, conditional on income. Under the condition stated by lemma 2, agents with higher risk aversion have lower savings than lower risk aversion agents, conditional on income. If the welfare weight grows (declines) with risk aversion, conditional on income, the covariance term turns out to be negative (positive), meaning the optimal tax is likely to be positive (negative).[15]

4.6 Optimal Non-linear Taxation of Capital Income

In this section, I explore an alternative regime in which the government applies a single, non-linear tax schedule to the expected level of capital income. With non-linear taxation, a timing issue arises under aggregate risk: individuals get their returns on savings in the second period, while the government sets marginal tax rates in the first period before returns are realised and observed, even though they should depend on them.[16] As setting the tax rates directly ex-post would be unrealistic given real-world tax policy, I assume the government optimises with respect to the expected level of

[15] This highlights a key distinction between unobservable preferences and unobservable investment ability as in Boadway and Spiritus (2024). When heterogeneity is driven by investment ability, the planner unequivocally targets redistribution toward low-ability agents, yielding a subsidy on the riskless return. When heterogeneity is driven by risk preferences, however, the direction of redistribution, and whether the optimal safe return tax is positive or negative, depends on the chosen welfare criteria normalising utilities across risk aversion types of agent.

[16] This was not an issue with linear taxation, as the tax rate on the excess return was the same regardless of the realised return.

capital income $\bar{y}(z,\theta)$ defined as

$$\bar{y}(z,\theta) \;\equiv\; \Big(r + s^*(\theta)\; \mathbb{E}[\tilde{r}^{exc}]\Big)\, a(z,\theta) \tag{28}$$

that is a deterministic function of the type of agent, since it uses the expected excess return $\mathbb{E}[\tilde{r}^{exc}]$ rather than its realization. Unlike non-linear taxes on realized returns, taxing expected capital income allows the government to commit to an ex-ante schedule, as tax revenues are not state-dependent. Similarly to the linear taxation case, the government knows the ex-ante distribution of types (without identifying them) and the market returns, and observes individual savings and the portfolio decomposition between risky and riskless assets. When the government commits ex ante to a non-linear schedule $T_k(\bar{y})$, applied to agents on the basis of their own expected capital income (28), the agent's budget constraint becomes

$$\tilde{c}_1(w,\theta) = \tilde{R}_p(\theta)\, a(w,\theta) - T_k\big(\bar{y}(w,\theta)\big); \qquad \tilde{R}_p = 1 + r + s(\theta)\, \tilde{r}^{exc}. \tag{29}$$

The FOCs for consumption and the share of risky assets are:

$$u'(c_0) = \mathbb{E}\left[u'(\tilde{c}_1)\left(\tilde{R}_p - T_k'\big(\bar{y}(w,\theta)\big)\cdot \frac{\partial \bar{y}}{\partial a}\right)\right],$$
$$\mathbb{E}\left[u'(\tilde{c}_1)\, \tilde{r}^{exc}\right] = T_k'\big(\bar{y}(w,\theta)\big)\cdot \mathbb{E}[\tilde{r}^{exc}]\cdot \mathbb{E}\left[u'(\tilde{c}_1)\right].$$

where $\partial\bar{y}/\partial a = r + s\; \mathbb{E}[\tilde{r}^{exc}] \equiv \bar{R}_p - 1$. Unlike the linear tax case, the nonlinear schedule introduces a wedge directly on the consumption-savings margin, since $\bar{y}$ depends on savings a and share of risky assets s.

The schedule $T_k(\bar{y})$ is applied to agents on the basis of their expected capital income $\bar{y}$, which depends on the agent-type. The relevant cumulative distribution given by the level-set integral of $f(z,\theta)$ over the region of the type-space Ω_z with expected capital incomes at or below $\bar{y}$, that is the level at which the tax perturbation takes place:

$$\int_{\{(z,\theta)\,:\,\bar{y}(z,\theta)\leq \bar{y}\}} f(z,\theta)\, d(z,\theta). \tag{30}$$

with corresponding density $\bar{h}(\bar{y}) \;=\; \frac{d}{d\bar{y}}\,\bar{H}(\bar{y})$, that is assumed to be well-behaved, with no bunching.

Then, I define the average social marginal welfare weight among agents whose expected capital income exceeds $\bar{y}$, via integration over the relevant sub-region of Ω_z:

$$\bar{G}(\bar{y}) \;\equiv\; \frac{\displaystyle\int_{\Omega_z} g(z,\theta)\, \mathbb{1}\{\bar{y}(z,\theta) > \bar{y}\}\, f(z,\theta)\, d(z,\theta)}{1 - \bar{H}(\bar{y})}. \tag{31}$$

Because $\bar{y}(z,\theta)$ generally depends on both z and θ, the region $\{(z,\theta) : \bar{y}(z,\theta) > \bar{y}\}$ pools together agents with different combinations of income and risk aversion that share the same (or higher) implied expected capital income. The following proposition presents the condition for the optimal marginal tax rate, obtained through a perturbation approach. The derivation is available in the appendix A.6.

Proposition 6. *Under a non-linear tax schedule on capital income, the optimal marginal tax rate is set by the following condition:*

$$\frac{T_k'(\bar{y})}{1 - T_k'(\bar{y})} = \frac{1}{\bar{y}\, \bar{\varepsilon}_{\bar{y},k}(\bar{y})} \left[\frac{\big(1 - \bar{H}(\bar{y})\big)\big(\alpha - \bar{G}(\bar{y})\big)}{\alpha\, \bar{h}(\bar{y})} - \frac{1}{\bar{h}(\bar{y})} \int_{\bar{y}}^{\infty} \eta_{\bar{y}}(\bar{y}')\, T_k'(\bar{y}')\, \bar{h}(\bar{y}')\, d\bar{y}' \right] \tag{32}$$

with $\bar{H}(\bar{y})$, $\bar{h}(\bar{y})$, $\bar{G}(\bar{y})$ defined by level-set integration over Ω_z as in (30)–(31), and $\bar{\varepsilon}_{\bar{y},k}(\bar{y})$ the composite elasticity averaged over the level set at $\bar{y}$ using the Dirac measure $\delta(\cdot)$ to restrict the integral to the level set $\{\bar{y}(z,\theta) = \bar{y}\}$:

$$\bar{\varepsilon}_{\bar{y},k}(\bar{y}) \;\equiv\; \frac{\displaystyle\int_{\Omega_z} \varepsilon_{\bar{y},k}(z,\theta) \cdot \bar{y}(z,\theta)\, \delta\big(\bar{y}(z,\theta) - \bar{y}\big)\, f(z,\theta)\, d(z,\theta)}{\bar{y}\, \bar{h}(\bar{y})}, \tag{33}$$

Condition (32) has a similar structure to the labour income tax formula. The term $\alpha - \bar{G}(\bar{y})$ captures the net social gain from raising marginal taxes on capital income in the band $(\bar{y}, \bar{y} + \Delta\bar{y})$ to finance the public good. The elasticity term $\bar{\varepsilon}_{\bar{y},k}$ and the hazard rate $\bar{h}(\bar{y})/\big(1 - \bar{H}(\bar{y})\big)$ capture the efficiency channel, driving marginal tax rates down. Notably, the elasticity $\bar{\varepsilon}_{\bar{y},k}$ is a composite of two distinct behavioural margins, savings a and portfolio share s:

$$\varepsilon_{\bar{y},k}(z,\theta) \;=\; \varepsilon_{a,k}(z,\theta) \;+\; \frac{s^*(\theta)\, \mathbb{E}[\tilde{r}^{exc}]}{r + s^*(\theta)\, \mathbb{E}[\tilde{r}^{exc}]} \cdot \varepsilon_{s,k}(z,\theta). \tag{34}$$

where the elasticity of the share of risky assets s is weighted by the term $\frac{s^*(\theta)\,\mathbb{E}[\tilde{r}^{exc}]}{r+s^*(\theta)\,\mathbb{E}[\tilde{r}^{exc}]}$, that is the share of expected capital income that comes from the risky-excess-return component. For simplicity, I assume a separable tax system in labour and capital income, such that condition (32) abstracts from the cross-base fiscal externalities that this tax imposes on the labour income base.[17]

Relative to the pair of linear taxes (τ_r, τ_k), nonlinear taxation of capital income, $T_k(\bar{y})$ distorts also portfolio decisions, alongside the savings margin, as the Domar-Musgrave neutrality result breaks down. This additional distortion can be welfare improving if the variation in welfare weights across risk aversion types, conditional on income, is significantly driven by differences in expected capital income due to portfolio decisions. Ultimately, this would be the case when risk preferences are widely dispersed within, and weakly correlated with, income levels.

Finally, non-linear taxation of capital income also changes the design of the optimal labour income tax schedule. The capital income fiscal externality terms in condition (22), previously evaluated at the linear rates τ_r and τ_k, would be replaced by the cross-base spillovers onto expected capital income. Specifically, a marginal increase in the labour tax induces behavioural responses in savings and portfolio shares that alter $\bar{y}$, generating a fiscal externality evaluated at the household-specific marginal capital tax rate $T_k'(\bar{y})$.

5 Concluding Remarks

This paper presents a theory of optimal provision and funding of a public good when individuals have heterogeneous preferences for risk and face aggregate risk in the economy. Under private provision, alongside the standard underprovision result, the Nash equilibrium entails an inefficient allocation of risk between private and public consumption, as individuals fail to internalize the insurance gains that the public good provides to agents with different risk preferences. These inefficiencies persist under

[17] A complete derivation would take into account that the change of the marginal tax on expected capital income induces responses in labour supply (analogous to the cross-substitution and income effects derived in Propositions 3 and 5), generating additional fiscal spillovers evaluated at $T'(z)$. However, formally including these terms would only make the tax formula more complex without changing the main intuition.

public provision, financed by a uniform linear tax on savings, when risk preferences are unobservable. Within a model with bi-dimensional heterogeneity – labour productivity and risk preferences – I then characterise the optimal tax system, in which each instrument plays a distinct role. The non-linear labour income tax redistributes across income levels while internalizing fiscal externalities on capital tax bases. The linear tax on the risky excess return performs a pure insurance function, aligning the volatility of the public good with a weighted average of private consumption volatility across risk-aversion types. The tax on the riskless return redistributes across risk-aversion types within income levels, with its sign depending on how welfare weights evolve along the distribution of risk preferences. As an alternative, non-linear taxation of expected capital income is considered: it gains screening power over unobserved risk preferences through the portfolio-share margin, at the cost of breaking the Domar–Musgrave neutrality of the linear excess-return tax. In all cases, the tax conditions are derived in terms of appropriately defined elasticities and welfare weights, and capture the key trade-off in this economy: redistributive objectives versus the efficiency costs of distorting labour supply, savings and portfolio decisions.

Several extensions are left for future research. First, individuals may differ not only in their risk preferences but also in their direct valuation of the public good; allowing for both sources of taste heterogeneity would help disentangle their respective roles in applications such as infrastructure or climate policy. Second, a general tax schedule conditioning jointly on labour and capital income could improve screening on the risk-aversion dimension, at the cost of solving a multidimensional problem.

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Appendix A

A.1 Proposition 1

We can decompose the social marginal rate of substitution $MRS_j^{\text{soc}} = \frac{\sum_k \omega_k \mathbb{E}\left[u_k'(\tilde{P})\tilde{R}_p^j\right]}{\omega_j \mathbb{E}\left[u_j'(\tilde{c}_1^j)\tilde{R}_p^j\right]}$ from condition (6) as follows:

$$\begin{aligned} MRS_j^{\text{soc}} &= \frac{\omega_j \mathbb{E}\left[u_j'(\tilde{P})\tilde{R}_p^j\right] + \sum_{k \neq j} \omega_k \mathbb{E}\left[u_k'(\tilde{P})\tilde{R}_p^j\right]}{\omega_j \mathbb{E}\left[u_j'\left(\tilde{c}_1^j\right)\tilde{R}_p^j\right]} \\ &= \underbrace{\frac{\mathbb{E}\left[u_j'(\tilde{P})\tilde{R}_p^j\right]}{\mathbb{E}\left[u_j'\left(\tilde{c}_1^j\right)\tilde{R}_p^j\right]}}_{MRS_j^{\text{r}}} + \underbrace{\frac{\sum_{k \neq j} \omega_k \mathbb{E}\left[u_k'(\tilde{P})\tilde{R}_p^j\right]}{\omega_j \mathbb{E}\left[u_j'\left(\tilde{c}_1^j\right)\tilde{R}_p^j\right]}}_{\Xi_j > 0}. \end{aligned}$$

The term Ξ_j is the externality: the risk-adjusted benefit that agent j 's contribution provides to all other agents $k \neq j$, normalised by agent j 's own marginal cost. Since $\omega_k > 0, u_k' > 0$, $\tilde{R}_p^j > 0$, and $\mathbb{E}\left[u_j'\left(\tilde{c}_1^j\right)\tilde{R}_p^j\right] > 0$ at an interior solution, every term in Ξ_j is strictly positive, so $\Xi_j > 0$ for all j. With a positive externality, the public good will be indeed under-provided in expectation at the Nash equilibrium.

A.2 Lemma 1

The optimal choice of $\tau(\theta_j)$ that maximises social welfare $SW = \sum_j \omega_j U_j$ gives the following optimality condition:

$$\mathbb{E}\left[\sum_k \omega_k \left(u_k'(\tilde{P})\right)\tilde{R}_p^j \left(1 - \frac{\tau(\theta_j)}{1-\tau(\theta_j)}\varepsilon_{a,\tau}(\theta_j) - \tau(\theta_j) \cdot \frac{\eta_{a,\tau}}{a_j}(\theta_j)\right)\right] - \omega_j \mathbb{E}\left[u_j'(\tilde{c}_1^j)\tilde{R}_p^j\right] = 0 \tag{A.1}$$

Define $p(\theta_j) = \frac{\mathbb{E}\left[\sum_k \omega_k \left(u_k'(\tilde{P})\right)\tilde{R}_p^j\right]}{\mathbb{E}\left[\omega_j u_j'(\tilde{c}_1^j)\tilde{R}_p\right]}$ as the ratio between the social welfare gain from an additional unit of public good for all types through a higher tax on agent-type j, and the private welfare cost for that agent j. Then, (A.22) can be rewritten as:

$$p(\theta_j)\left[1 - \frac{\tau(\theta_j)}{1-\tau(\theta_j)}\varepsilon_{a,\tau}(\theta_j) - \tau \cdot \frac{\eta_{a,\tau}}{a_j}(\theta_j)\right] = 1. \tag{A.2}$$

Isolating $\frac{\tau}{1-\tau}$ allows to get condition (11).

A.3 Proposition 3

Consider a small reform in which the marginal tax rate is increased by $d\tau_l$ in a small earnings band $(z, z+\Delta z)$, with no change elsewhere in the earnings distribution. This perturbation will have the following effects. First, there is a mechanical effect raising tax revenues:

$$M = \Delta z\, d\tau_l (1 - H(z))\alpha. \tag{A.3}$$

where α the social welfare gain per unit of public good provision, normalized by the marginal value of public funds λ. For each state of nature, I also define the state-contingent marginal value of public funds $\tilde{\alpha}$:

$$\tilde{\alpha} = \frac{1}{\lambda} \int_{\Omega_z} \omega(z,\theta)\, u'(\tilde{P}; z, \theta) \cdot f(z,\theta)\, d(z,\theta), \qquad \alpha \equiv \mathbb{E}[\tilde{\alpha}].$$

Second, there are behavioural effect due to substitution and income effects. The substitution effect involves only the taxpayers in the band $(z, z+\Delta z)$, while the income effect involves everyone above z.

$$\begin{aligned} B_1 &= -\alpha\, h(z)\Delta z\, d\tau_l\, \bar{\varepsilon}_{z,l} \cdot \frac{z\,T'(z)}{1-T'(z)} - h(z)\Delta z\, d\tau_l \left[\frac{\alpha\, \bar{\varepsilon}^a_{a,l} \cdot \tau_r r + \bar{\varepsilon}^a_{k,l} \cdot \tau_k\ \mathbb{E}[\tilde{\alpha}\tilde{r}^{\text{exc}}]}{1-T'(z)}\right] \\ &= -\alpha\, h(z)\Delta z\, d\tau_l \left[\bar{\varepsilon}_{z,l} \cdot \frac{z\,T'(z)}{1-T'(z)} + \frac{\bar{\varepsilon}^a_{a,l} \cdot \tau_r r + \bar{\varepsilon}^a_{k,l} \cdot \tau_k\ \mathbb{E}[\tilde{\alpha}\tilde{r}^{\text{exc}}]/\alpha}{1-T'(z)}\right] \end{aligned} \tag{A.4}$$

where $\bar{\varepsilon}^a_{a,l} = \int \varepsilon_{a,l}(z,\theta) \cdot a(z,\theta) f(\theta|z) d\theta$. Then, the term with the income effects:

$$B_2 = \alpha\, \Delta z\, d\tau_l \int_z^\infty \Big(\bar{\eta}_l(z')\, T'(z') - \bar{\eta}^a_l(z')\, \tau_r r - \frac{\mathbb{E}[\tilde{\alpha}\tilde{r}^{\text{exc}}]}{\alpha} \cdot \bar{\eta}^k_l\,(z')\, \tau_k\Big) h(z') dz' \tag{A.5}$$

where $\bar{\eta}_l(z') = \int_\theta \eta_l(z',\theta) \cdot f(\theta|z')\, d\theta$ is the income effect averaged across risk aversion types at income level z' with $\eta_l(z',\theta) \equiv -\partial z'(\theta)/\partial I\big|_{T'(z')}$, and I non-labour income. Moreover, $\bar{\eta}^a_l(z') = \int_\theta \eta_a(z',\theta) \cdot a(z',\theta) \cdot f(\theta|z')\, d\theta$ with $\eta_a(z',\theta) \equiv \partial a(z,\theta)/\partial I\big|_{T'(z')}$, and $\bar{\eta}^k_l\,(z') = \int \eta_k\,(z',\theta) \cdot s(\theta) a\,(z',\theta) \cdot f\,(\theta \mid z')\, d\theta$ with $\eta_k(z',\theta) \equiv \partial(s \cdot a)/\partial I\big|_{T'(z')}$. Third, there is a welfare effect due to the utility loss generated by the tax perturbation.

$$W = -d\tau_l \Delta z \int_z^\infty \bar{g}(z')\, h(z') d\, z' = -d\tau_l \Delta z \cdot [1 - H(z)] \overline{G}(z). \tag{A.6}$$

where $\overline{G}(z) = \frac{\int_z^\infty \overline{g}(z')\,h(z')dz'}{1-H(z)}$ is the average social marginal welfare weight above income level z, and $\overline{g}(z') = \int g(z',\theta)\cdot f(\theta|z')\,d\theta$.

Putting everything together we have:

$$\frac{T'(z)}{1-T'(z)} = \frac{1}{z\,\overline{\varepsilon}_{z,l}}\left[\frac{(1-H(z))(\alpha-\overline{G}(z))}{\alpha\,h(z)} - \frac{\overline{\varepsilon}^a_{a,l}\cdot\tau_r r + \overline{\varepsilon}^a_{k,l}\cdot\tau_k\ \mathbb{E}[\tilde{\alpha}\tilde{r}^{\text{exc}}]/\alpha}{1-T'(z)}\right.$$
$$\left. + \frac{1}{h(z)}\int_z^\infty \left(\overline{\eta}_l(z')\,T'(z') - \overline{\eta}^a_l(z')\,\tau_r r - \frac{\mathbb{E}[\tilde{\alpha}\tilde{r}^{\text{exc}}]}{\alpha}\cdot\overline{\eta}^k_l\,(z')\,\tau_k\right)h(z')dz'\right] \quad \text{(A.7)}$$

As for the intercept, a small perturbation $d\tau_0$ generates a mechanical effect $R = \lambda\cdot\alpha\,d\tau_0$ and a welfare effect $W = -d\tau_0\int_{\Omega_z}\omega(z,\theta)U_c(z,\theta)\cdot f(z,\theta)d(z,\theta)$. Setting $R+W=0$, and using (21), gives the condition

$$\alpha = \int_{\Omega_z} g(z,\theta)\cdot f(z,\theta)d(z,\theta).$$

Given the definition of α, and $g(z,\theta)$, this condition can be also rewritten as

$$\int_{\Omega_z}\mathbb{E}\left[\omega(z,\theta)u'(\tilde{P};z,\theta)\right]f(z,\theta)\,d(z,\theta) = \int_{\Omega_z}\mathbb{E}\left[\omega(z,\theta)U_c(z,\theta)\right]f(z,\theta)\,d(z,\theta) \quad \text{(A.8)}$$

Finally, at the joint optimum, when the excess return is set optimally, the fiscal externality involving the excess return vanishes (see proof of corollary 2 in appendix A.7).

A.4 Proposition 4

Consider the welfare effect of a marginal change in the tax rate τ_k. By FOC (16), the private welfare cost is zero. Hence, at the optimum, the welfare change is:

$$\mathbb{E}\left[\int_{\Omega_z}\omega(z,\theta)u'(\tilde{P};z,\theta)f(z,\theta)\,d(z,\theta)\cdot\int_{\Omega_z}f(z,\theta)s^*_j(\tilde{r}^{exc})\tilde{r}^{exc}a(z,\theta)\,d(z,\theta)\right]$$
$$+\tau_k\mathbb{E}\left[\int_{\Omega_z}\omega(z,\theta)u'(\tilde{P};z,\theta)f(z,\theta)\,d(z,\theta)\cdot\int_{\Omega_z}f(z,\theta)\frac{\partial s^*_j}{\partial\tau_k}\tilde{r}^{exc}a(z,\theta)\,d(z,\theta)\right] = 0. \quad \text{(A.9)}$$

Using the fact that by optimal portfolio choice $\frac{\partial s}{\partial \tau_k} = \frac{s}{1-\tau_k}$, we can simplify it to:

$$\mathbb{E}\left[\int_{\Omega_z} \omega(z,\theta)u'(\tilde{P};z,\theta)f(z,\theta)\,d(z,\theta)\cdot\int_{\Omega_z} f(z,\theta)s_j^*(\tilde{r}^{exc})\tilde{r}^{exc}a(z,\theta)\,d(z,\theta)\right]=0 \tag{A.10}$$

This can be further simplified by moving $\tilde{r}^{exc}$ to the first integral, and dropping the second integral, which is a positive constant.

$$\mathbb{E}\left[\tilde{r}^{exc}\int_{\Omega_z} \omega(z,\theta)u'(\tilde{P};z,\theta)f(z,\theta)\,d(z,\theta)\right]=0 \tag{A.11}$$

This condition states that at the optimum, the expected product of the aggregate excess return and the welfare-weighted marginal utility of the public good is zero.

Furthermore, using the optimality condition (A.8) for τ_0, the condition can be rewritten as follows:

$$\begin{aligned}\int_{\Omega_z} \omega(z,\theta)\mathrm{Cov}\left(\tilde{r}^{exc},u'(\tilde{P};z,\theta)\right)f(z,\theta)\,d(z,\theta)\\ =\int_{\Omega_z} \omega(z,\theta)\mathrm{Cov}\left(\tilde{r}^{exc},u'(\tilde{c}_1;z,\theta)\right)f(z,\theta)\,d(z,\theta)\end{aligned} \tag{A.12}$$

Moreover, the optimal tax rate is positive but strictly lower than 100%. Only $0\% < \tau_k < 100\%$ ensures that the left and right hand side of the optimality condition are non-zero terms with the same sign.

A.5 Proposition 5

A small reform $d\tau_r$ generates the following effects. First, the mechanical effect is

$$M = \alpha\, r\, A\, d\tau_r.$$

where $A=\int_{\Omega_z} a(z,\theta)f(z,\theta)\,d(z,\theta)$ is aggregate savings, and α is the social welfare gain per unit of public good, normalised by the marginal value of public funds. Second, there is a welfare effect related to the utility loss of raising the tax:

$$W = -r\,d\tau_r\int_{\Omega_z} g(z,\theta)\cdot a(z,\theta)\cdot f(z,\theta)\,d(z,\theta).$$

Then, the reform will generate behavioural effects through savings and labour income. The term B_1 refers to the substitution effects:

$$B_1 = -\left[\alpha \frac{\tau_r r}{1-\tau_r} \int_{\Omega_z} \varepsilon_{a,r}(z,\theta) \cdot a(z,\theta) \cdot f(z,\theta)\, d(z,\theta) - \frac{\alpha}{1-\tau_r} \int_{\Omega_z} \varepsilon_{l,r}(z,\theta) z \cdot T'(z) \cdot f(z,\theta)\, d(z,\theta)\right] d\tau_r.$$

where $\varepsilon_{a,r}(z,\theta)$ is the compensated elasticity of riskless savings with respect to the after-tax riskless return $r(1-\tau_r)$; and $\varepsilon_{l,r}(z,\theta)$ is the cross compensated elasticity of labour earnings with respect to the after-tax riskless return $r(1-\tau_r)$. B_2 refers to the income effects:

$$B_2 = \left[\alpha r \int_{\Omega_z} \eta_l(z,\theta) \cdot T'(z) \cdot a(z,\theta) \cdot f(z,\theta)\, d(z,\theta) - \alpha\, \tau_r r \int_{\Omega_z} \eta_a(z,\theta) \cdot a(z,\theta) \cdot f(z,\theta)\, d(z,\theta)\right] d\tau_r.$$

where $\eta_l(z,\theta)$ is the income effect on labour earnings; and $\eta_a(z,\theta)$ is the income effect on riskless savings. The risky-capital income fiscal externality is omitted as it drops out by Corollary 2.

Hence, introducing a tax or subsidy on the normal return (riskless capital income) is welfare enhancing when the following condition is satisfied:

$$\left.\frac{\partial SW}{\partial \tau_r}\right|_{\tau_r=0,\tau_k^*} = \alpha A - \int_{\Omega_z} g(z,\theta) \cdot a(z,\theta) \cdot f(z,\theta)\, d(z,\theta) + \alpha \int_{\Omega_z} T'(z) \left[\eta_l(z,\theta) \cdot a(z,\theta) - \varepsilon_{l,r}(z,\theta) \cdot z\right] f(z,\theta)\, d(z,\theta) \neq 0 \quad \text{(A.13)}$$

where $A = \int_{\Omega_z} a(z,\theta) \cdot f(z,\theta)\, d(z,\theta)$ is aggregate savings. This can be rewritten as

$$\left.\frac{\partial SW}{\partial \tau_r}\right|_{\tau_r=0,\tau_k^*} = -\operatorname{Cov}(g(z,\theta), a(z,\theta)) + \alpha \int_{\Omega_z} T'(z) \left[\eta_l(z,\theta) \cdot a(z,\theta) - \varepsilon_{l,r}(z,\theta) \cdot z\right] f(z,\theta)\, d(z,\theta) \neq 0 \quad \text{(A.14)}$$

after noticing that

$$\begin{aligned}\mathrm{Cov}(g(z,\theta),a(z,\theta)) \equiv & \int_{\Omega_z} g(z,\theta)a(z,\theta)f(z,\theta)d(z,\theta) \\ & - \underbrace{\left(\int_{\Omega_z} g(z,\theta)f(z,\theta)d(z,\theta)\right)}_{=\alpha \text{ by } (23)} \underbrace{\left(\int_{\Omega_z} a(z,\theta)f(z,\theta)d(z,\theta)\right)}_{\equiv A}\end{aligned} \tag{A.15}$$

A.6 Proposition 6

Consider a small reform: raise $T_k'(\cdot)$ by $d\tau$ over a narrow band $(\bar{y}, \bar{y}+\Delta\bar{y})$ of the distribution of expected capital income $\bar{y} = a(z,\theta)[r + s(\theta)\,\mathbb{E}[\tilde{r}^{exc}]]$, holding the schedule unchanged elsewhere.

Mechanical effect: every agent with $\bar{y}(z,\theta) > \bar{y}$ pays $d\tau \cdot \Delta\bar{y}$ in extra tax. Using (30), the mass of such households is $1 - \bar{H}(\bar{y})$, so the mechanical revenue effect, valued at the social marginal value of public funds α, is

$$M = \Delta\bar{y}\, d\tau \left(1 - \bar{H}(\bar{y})\right) \alpha. \tag{A.16}$$

Behavioural effect: only households with $\bar{y}(z,\theta) \in (\bar{y}, \bar{y}+\Delta\bar{y})$ face a marginal-rate change, and they adjust both the share of risky assets s and savings a . Define the compensated elasticities, type by type,

$$\varepsilon_{s,k}(z,\theta) \equiv \frac{\partial s^*}{\partial(1-T_k')} \cdot \frac{1-T_k'}{s^*}\bigg|_U, \qquad \varepsilon_{a,k}(z,\theta) \equiv \frac{\partial a}{\partial(1-T_k')} \cdot \frac{1-T_k'}{a}\bigg|_U. \tag{A.17}$$

The induced elasticity (A.18) of $\bar{y}$ itself, combining both margins through (28), is

$$\varepsilon_{\bar{y},k}(z,\theta) \;=\; \varepsilon_{a,k}(z,\theta) \;+\; \frac{s^*(\theta)\;\mathbb{E}[\tilde{r}^{exc}]}{r + s^*(\theta)\,\mathbb{E}[\tilde{r}^{exc}]} \cdot \varepsilon_{s,k}(z,\theta). \tag{A.18}$$

Averaging over the (possibly pooled) types within the affected band using the same direct level-set integration as in (31),

$$\bar{\varepsilon}_{\bar{y},k}(\bar{y}) \;\equiv\; \frac{\displaystyle\int_{\Omega_z} \varepsilon_{\bar{y},k}(z,\theta) \cdot \bar{y}(z,\theta)\, \delta\big(\bar{y}(z,\theta) - \bar{y}\big)\, f(z,\theta)\, d(z,\theta)}{\bar{y}\, \bar{h}(\bar{y})}, \tag{A.19}$$

where $\delta(\cdot)$ is the Dirac measure restricting the integral to the level set $\{\bar{y}(z,\theta)=\bar{y}\}$, the substitution effect is

$$B_1 = -\alpha\, \bar{h}(\bar{y})\, \Delta\bar{y}\, d\tau \cdot \bar{\varepsilon}_{\bar{y},k}(\bar{y}) \cdot \frac{\bar{y}\, T_k'(\bar{y})}{1-T_k'(\bar{y})}. \tag{A.20}$$

Income effect: agents above the band, $\bar{y}(z,\theta) > \bar{y}+\Delta\bar{y}$, experience an income effect from the higher average tax paid. Define $\eta_{\bar{y}}(\bar{y}')$ as the income effect on $\bar{y}$ at threshold $\bar{y}'$, averaged over the level set $\{\bar{y}(z,\theta)=\bar{y}'\}$ exactly as above. The income effect term is

$$B_2 = -\alpha\, \Delta\bar{y}\, d\tau \int_{\bar{y}'} \eta_{\bar{y}}(\bar{y}')\, T_k'(\bar{y}')\, \bar{h}(\bar{y}')\, d\bar{y}'. \tag{A.21}$$

Welfare effect: using (31), the utility loss to households above the band is

$$W = -d\tau\, \Delta\bar{y}\, \big(1-\bar{H}(\bar{y})\big)\, \bar{G}(\bar{y}). \tag{A.22}$$

Setting $M + B_1 + B_2 + W = 0$ and dividing through by $\alpha\, \bar{h}(\bar{y})\, \Delta\bar{y}\, d\tau$ gives the optimality condition

$$\frac{T_k'(\bar{y})}{1-T_k'(\bar{y})} = \frac{1}{\bar{y}\, \bar{\varepsilon}_{\bar{y},k}(\bar{y})} \left[\frac{\big(1-\bar{H}(\bar{y})\big)\big(\alpha - \bar{G}(\bar{y})\big)}{\alpha\, \bar{h}(\bar{y})} - \frac{1}{\bar{h}(\bar{y})} \int_{\bar{y}}^{\infty} \eta_{\bar{y}}(\bar{y}')\, T_k'(\bar{y}')\, \bar{h}(\bar{y}')\, d\bar{y}' \right] \tag{A.23}$$

with $\bar{H}(\bar{y})$, $\bar{h}(\bar{y})$, $\bar{G}(\bar{y})$ defined by direct level-set integration over Ω_z as in (30)–(31), and $\bar{\varepsilon}_{\bar{y},k}(\bar{y})$ the composite elasticity (A.18) averaged over the level set at $\bar{y}$.

A.7 Corollary 2

In the perturbation approach derivation of condition (22), the compensated and income effect terms involving the risky excess return can be grouped as follows:

$$-\frac{\mathbb{E}[\tilde{\alpha}\, \tilde{r}^{\text{exc}}]}{\alpha} \Delta z\, d\tau_l \left[\frac{h(z) \cdot \bar{\varepsilon}^{a}_{k,l} \cdot \tau_k}{1-T'(z)} + \int_z^{\infty} \Big(\bar{\eta}_l^k\, (z')\, \tau_k \Big) h(z') dz' \right] = 0$$

This term is equal to zero as $\mathbb{E}[\tilde{\alpha}\, \tilde{r}^{\text{exc}}] = \frac{1}{\lambda}\, \mathbb{E}\left[\tilde{r}^{\text{exc}} \cdot \int_{\Omega_z} \omega(z,\theta)\, u'(\tilde{P}; z,\theta) \cdot f\,(z,\theta)\, d(z,\theta) \right] = 0$ by the excess return tax condition (24).

A.8 Corollary 3

Exploting the FOC (16), the private welfare cost is zero, and the optimality condition of τ_k^0 for agents of type (z, θ) reads:

$$\mathbb{E}\left[f(z,\theta)s^*(\theta)(\tilde{r}^{exc})\tilde{r}^{exc}a(z,\theta)\cdot\int_{\Omega_z}\omega(z,\theta)u'(\tilde{P};z,\theta)f(z,\theta)\,d(z,\theta)\right]$$
$$+\tau_k^0\mathbb{E}\left[f(z,\theta)\frac{\partial s^*(\theta)}{\partial\tau_k^0}\tilde{r}^{exc}a(z,\theta)\cdot\int_{\Omega_z}\omega(z,\theta)u'(\tilde{P};z,\theta)f(z,\theta)\,d(z,\theta)\right]=0 \quad \text{(A.24)}$$

Using the fact that by optimal portfolio choice $\frac{\partial s}{\partial \tau_k} = \frac{s}{1-\tau_k}$, we can simplify it to:

$$\mathbb{E}\left[f(z,\theta)s^*(\theta)(\tilde{r}^{exc})\tilde{r}^{exc}a(z,\theta)\cdot\int_{\Omega_z}\omega(z,\theta)u'(\tilde{P};z,\theta)f(z,\theta)\,d(z,\theta)\right]=0 \quad \text{(A.25)}$$

This can be further simplified by moving $\tilde{r}^{exc}$ to the first integral, and dropping the second integral, which is a positive constant.

$$\mathbb{E}\left[\tilde{r}^{exc}\int_{\Omega_z}\omega(z,\theta)u'(\tilde{P};z,\theta)f(z,\theta)\,d(z,\theta)\right]=0 \quad \text{(A.26)}$$

The resulting condition coincides with the condition for a unique τ_k, implying that deviating from the unique τ_k does not further raise welfare.

$\square$

Appendix B

Throughout this appendix, lowercase returns denote log returns: $r \equiv \log R$ $\tilde{r}_p \equiv \log \tilde{R}_p$, so that $\tilde{r}_p - r = \log(\tilde{R}_p/R)$.

B.1 Optimal Portfolio Share of Risky Assets

To derive formula (18), I follow Campbell and Viceira (2002). The portfolio return factor is $\tilde{R}_p = (1 + r(1-\tau_r)) + s(R^e - R)(1-\tau_k)$, or

$$\frac{\tilde{R}_p}{R} = 1 - \frac{r}{R}\tau_r + s\Big(\frac{R^e}{R} - 1\Big)(1-\tau_k).$$

Taking logs we get

$$\tilde{r}_p - r = f(r^e - r) = \log\Big(1 - \frac{r}{R}\tau_r + s(e^{r^e - r} - 1)(1-\tau_k)\Big). \tag{B.1}$$

It's possible to derive an approximation of the portfolio excess return using a second-degree Taylor's expansion around the point $r^e - r = 0$:

$$\tilde{r}_p - r = \log\Big(1 - \frac{r}{R}\tau_r\Big) + \frac{s(1-\tau_k)R}{R - r\tau_r}\tilde{\varphi} + \frac{1}{2}\frac{s(1-\tau_k)R}{R - r\tau_r}\Big(1 - \frac{s(1-\tau_k)R}{R - r\tau_r}\Big)\sigma_r^2. \tag{B.2}$$

where $\tilde{\varphi} := \tilde{r}^e - r$. Then, we can write the expected utility at time 0 from consumption in period 1 as follows:

$$\begin{aligned}
\mathbb{E}_0(u(c_1)) &\approx (1-\theta)^{-1}\,\mathbb{E}_0\Big[\Big(a_0 e^r e^{\tilde{r}_p - r}\Big)^{1-\theta}\Big] \qquad\qquad (B.3)\\
&\approx (1-\theta)^{-1}\,\mathbb{E}_0\Big[\Big(a_0 e^r e^{\log\left(1-\frac{r}{R}\tau_r\right)+\frac{s(1-\tau_k)R}{R-r\tau_r}\tilde{\varphi}+\frac{s(1-\tau_k)R}{R-r\tau_r}\left(1-\frac{s(1-\tau_k)R}{R-r\tau_r}\right)\frac{\sigma_r^2}{2}}\Big)^{1-\theta}\Big]\\
&\approx (1-\theta)^{-1}\Big[a_t e^r\Big(1 - \frac{r}{R}\tau_r\Big)\Big]^{1-\theta} \underbrace{e^{(1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}\left(1-\frac{s(1-\tau_k)R}{R-r\tau_r}\right)\frac{\sigma_r^2}{2}}\,\mathbb{E}_0\Big[e^{(1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}\tilde{\varphi}}\Big]}_{\text{Excess return component}}
\end{aligned}$$

where the term $(1-\theta)^{-1}\Big[a_t e^r\Big(1-\frac{r}{R}\tau_r\Big)\Big]^{1-\theta}$ is constant. Because the constant prefactor $(1-\theta)^{-1}$ and the exponent multiplier $(1-\theta)$ carry the same sign in (B.3), their product is strictly positive for all $\theta > 0, \theta \neq 1$. Therefore, maximizing expected utility is equivalent in all cases to maximizing the "excess return component". Notice that the term inside the expectation can be rewritten as follows when the return is log-normally distributed:

$$(1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}\tilde{\varphi} \sim \mathcal{N}\Big((1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}(\varphi-\sigma_r^2/2), (1-\theta)^2\Big(\frac{s(1-\tau_k)R}{R-r\tau_r}\Big)^2\sigma_r^2\Big).$$

where $\varphi = \overline{r}^e - r$. Hence, taking the log of $\mathbb{E}_0\left[e^{(1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}\tilde{\varphi}}\right]$, we get:

$$\log \mathbb{E}_0\left[e^{(1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}\tilde{\varphi}}\right] = (1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}(\varphi-\sigma_r^2/2) + \Big(\frac{s(1-\tau_k)R}{R-r\tau_r}\Big)^2\frac{\sigma_r^2}{2}(1-\theta)^2.$$

Then, we can compute the log of the excess return component:

$$\begin{aligned}
\log(\text{Exc}) = (1-\theta)&\frac{s(1-\tau_k)R}{R-r\tau_r}\Big(1-\frac{s(1-\tau_k)R}{R-r\tau_r}\Big)\frac{\sigma_r^2}{2} + (1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}\Big(\varphi-\frac{\sigma_r^2}{2}\Big) \\
&+ (1-\theta)^2\left(\frac{s(1-\tau_k)R}{R-r\tau_r}\right)^2\frac{\sigma_r^2}{2} \\
&= (1-\theta)\left[\frac{s(1-\tau_k)R}{R-r\tau_r}\varphi - \theta\left(\frac{s(1-\tau_k)R}{R-r\tau_r}\right)^2\frac{\sigma_r^2}{2}\right]
\end{aligned} \tag{B.4}$$

Hence, it is possible to maximize expected utility by maximizing the terms multiplying $(1-\theta)$ in (B.4):

$$\max_s \frac{s(1-\tau_k)R}{R-r\tau_r}\varphi - \theta\Big(\frac{s(1-\tau_k)R}{R-r\tau_r}\Big)^2\frac{\sigma_r^2}{2},$$

with first order condition:

$$s_j^* = \frac{\varphi}{\theta_j\sigma_r^2}\cdot\frac{(R-r\tau_r)}{R\,(1-\tau_k)},$$

where $\varphi = \overline{r}^e - r$.

B.2 First-period Consumption

From the FOC of first period consumption, we can rewrite c_0 as follows

$$\begin{aligned}
c_0^{-\theta} &= \mathbb{E}\left[\tilde{R}_p^{1-\theta}\,(z-c_0)^{-\theta}\right] \\
c_0 &= \frac{z}{1+\mathbb{E}\left[\tilde{R}_p^{1-\theta}\right]^{1/\theta}}
\end{aligned}$$

For simplicity, we omit labour income taxation in the derivation, without loss of generality. Next, I derive an expression for $\mathbb{E}\left[\tilde{R}_p^{1-\theta}\right]$ and c_0 With $\tilde{r}_p = \log\tilde{R}_p \sim$

$\mathcal{N}\left(\bar{r}_p - \sigma_p^2/2, \sigma_p^2\right)$ we know as a fact the following holds

$$\begin{aligned}
\log \mathbb{E}\left[\tilde{R}_p\right] &= \mathbb{E}\left[\log \tilde{R}_p\right] + \sigma_p^2/2 \\
&= \bar{r}_p - \sigma_p^2/2 + \sigma_p^2/2 = \bar{r}_p
\end{aligned}$$

Using the previous fact and since $\log \tilde{R}_p^{1-\theta} = (1-\theta)\log \tilde{R}_p$, the following also holds

$$\begin{aligned}
\log \mathbb{E}\left[\tilde{R}_p^{1-\theta}\right] &= (1-\theta)\mathbb{E}\left[\log \tilde{R}_p\right] + (1-\theta)^2\sigma_p^2/2 \\
&= (1-\theta)\left(\bar{r}_p - \sigma_p^2/2\right) + (1-\theta)^2\sigma_p^2/2 \\
&= (1-\theta)\bar{r}_p - \frac{(1-\theta)\sigma_p^2}{2}(1-(1-\theta)) \\
&= (1-\theta)\bar{r}_p - \frac{\theta(1-\theta)\sigma_p^2}{2}
\end{aligned}$$

Taking the exponential of both LHS and RHS

$$\mathbb{E}\left[\tilde{R}_p^{1-\theta}\right] = \exp\left[(1-\theta)\bar{r}_p - \theta(1-\theta)\sigma_p^2/2\right]$$

Elevating that to the exponent $1/\theta$, it follows that

$$\mathbb{E}\left[\tilde{R}_p^{1-\theta}\right]^{1/\theta} = \exp\left[\left(\theta^{-1} - 1\right)\bar{r}_p - (1-\theta)\sigma_p^2/2\right]$$

Now we want to express $\bar{r}_p$ and σ_p^2 in terms of the risky asset. Define $\kappa = \frac{s(1-\tau_k)R}{R - r\tau_r}$.
From Campbell and Viceira (2002), given one risky asset in the portfolio $\tilde{r}_e \sim \mathcal{N}\left(\bar{r}_e - \sigma_r^2/2, \sigma_r^2\right)$, log excess portfolio return is approximately

$$\tilde{r}_p - r = \log\left(1 - \frac{r}{R}\tau_r\right) + \kappa\tilde{\varphi} + \frac{1}{2}\kappa\Big(1-\kappa\Big)\sigma_r^2$$

where $\tilde{\varphi} := \tilde{r}^e - r$. Then, expected excess return can be written as:

$$\mathbb{E}\left[\tilde{r}_p - r\right] = \log\left(1 - \frac{r\tau_r}{R}\right) + \kappa\left(\varphi - \frac{\sigma_r^2}{2}\right) + \frac{1}{2}\kappa\Big(1-\kappa\Big)\sigma_r^2 = \log\left(1 - \frac{r\tau_r}{R}\right) + \kappa\varphi - \frac{\kappa^2\sigma_r^2}{2}$$

with $\varphi := \bar{r}^e - r$ and variance $\sigma_p^2 = \left(\frac{s(1-\tau_k)R}{R-r\tau_r}\right)^2 \sigma_r^2$. We also know as a fact that

$$\begin{aligned}\log \mathbb{E}\left[\left(\tilde{R}_p/R\right)\right] &= \mathbb{E}\left[\log\left(\tilde{R}_p/R\right)\right] + \sigma_p^2/2 \\ &= \log\left(1 - \frac{r\tau_r}{R}\right) + \frac{s(1-\tau_k)R}{R - r\tau_r}\varphi\end{aligned}$$

so that we have

$$\log \mathbb{E}[\tilde{R}_p] = \bar{r}_p = r + \log\left(1 - \frac{r\tau_r}{R}\right) + \frac{s(1-\tau_k)R}{R - r\tau_r}\varphi$$

Recall also we have an approximated solution for s derived in appendix (B.1). So we can finally rewrite the expression for $\mathbb{E}\left[\tilde{R}_p^{1-\theta}\right]^{1/\theta}$ substituting in the expressions for s and $\bar{r}_p$:

$$\begin{aligned}\mathbb{E}\left[\tilde{R}_p^{1-\theta}\right]^{1/\theta} &= \exp\left[\left(\theta^{-1} - 1\right)\bar{r}_p - (1-\theta)\sigma_p^2/2\right] \\ &= \exp\left[\left(\theta^{-1} - 1\right)\left(r + \log\left(1 - \frac{r\tau_r}{R}\right) + \frac{s(1-\tau_k)R}{R - r\tau_r}\varphi\right) - (1-\theta)\sigma_p^2/2\right] \\ &= \left(1 - \frac{r\tau_r}{R}\right)^{\frac{1}{\theta}-1} \exp\left[\left(\theta^{-1} - 1\right)\left(r + \frac{s(1-\tau_k)R}{R - r\tau_r}\varphi\right) - (1-\theta)\sigma_p^2/2\right] \\ &= \left(1 - \frac{r\tau_r}{R}\right)^{\frac{1}{\theta}-1} \exp\left\{\left(\theta^{-1} - 1\right)\left(r + \frac{\varphi^2}{\theta\sigma_r^2}\right) - (1-\theta)\frac{\varphi^2}{2\theta^2\sigma_r^2}\right\} \equiv \Lambda(\theta)\end{aligned}$$

Therefore, we can rewrite first period consumption $c_0(z,\theta) = \frac{z}{1+\Lambda(\theta)}$ where

$$\Lambda(\theta) \equiv \left(1 - \frac{r\tau_r}{R}\right)^{\frac{1}{\theta}-1} \exp\left\{\left(\theta^{-1} - 1\right)\left(r + \frac{\varphi^2}{\theta\sigma_r^2}\right) - (1-\theta)\frac{\varphi^2}{2\theta^2\sigma_r^2}\right\}$$

and $\varphi \equiv \bar{r}^e - r > 0$ denoting the risk premium.

B.3 Savings behaviour Across Risk Aversion Types

For any $\tau_r \in [0,1)$, $c_0(z,\theta)$ is strictly increasing in θ, and savings $a(z,\theta) \equiv z - c_0(z,\theta)$ are strictly decreasing in θ, for all $\theta \in (0,2)$.

Proof. Since c_0 is a strictly decreasing function of Λ for $\Lambda > 0$,

$$\operatorname{sign}\left(\frac{\partial c_0}{\partial\theta}\right) = -\operatorname{sign}\left(\frac{d\Lambda}{d\theta}\right). \tag{B.5}$$

Let $\ell(\theta) \equiv \ln \Lambda(\theta)$ and $B \equiv \ln\left(1 - \frac{r\tau_r}{R}\right) \leq 0$. Then

$$\begin{aligned} \ell(\theta) &= (\theta^{-1} - 1)(r + B) + \frac{\varphi^2}{\sigma_r^2}\left[\frac{\theta^{-1} - 1}{\theta} - \frac{1-\theta}{2\theta^2}\right] \\ &= (1-\theta)\left[\frac{r+B}{\theta} + \frac{\varphi^2}{2\sigma_r^2\theta^2}\right] \end{aligned} \tag{B.6}$$

Differentiating (B.6) with respect to θ:

$$\begin{aligned} \frac{d\ell}{d\theta} &= -\left[\frac{r+B}{\theta} + \frac{\varphi^2}{2\sigma_r^2\theta^2}\right] + (1-\theta)\left[-\frac{r+B}{\theta^2} - \frac{\varphi^2}{\sigma_r^2\theta^3}\right] \\ &= -\frac{r+B}{\theta} - \frac{\varphi^2}{2\sigma_r^2\theta^2} - \frac{(1-\theta)(r+B)}{\theta^2} - \frac{(1-\theta)\varphi^2}{\sigma_r^2\theta^3} \quad = -\frac{r+B}{\theta^2} - \frac{(2-\theta)\,\varphi^2}{2\sigma_r^2\theta^3}. \end{aligned} \tag{B.7}$$

where the last line is obtained by collecting the $(r+B)$ terms and the φ^2 terms. Now, I analyse the two terms of (B.7).

Term 1. Since $\tau_r < R = 1 + r$ for any $\tau_r \in [-1, 1]$, we have $\frac{r\tau_r}{R} < r$, and the log bound $\ln(1-x) > -\frac{x}{1-x}$ for $x \in (0, 1)$ gives

$$B > -\frac{r\tau_r/R}{1 - r\tau_r/R} \quad \Longrightarrow \quad r + B > r - \frac{r\tau_r/R}{1 - r\tau_r/R} > 0$$

for any admissible $\tau_r \in [-1, 1]$ and $r > 0$. Hence $-\frac{r+B}{\theta^2} < 0$.

Term 2. For $\theta \in (0, 2)$, $(2 - \theta) > 0$, and $\varphi^2, \sigma_r^2, \theta^3 > 0$, so $-\frac{(2-\theta)\varphi^2}{2\sigma_r^2\theta^3} < 0$.

Both terms in (B.7) are strictly negative for all $\theta \in (0, 2)$, so $\frac{d\ell}{d\theta} < 0$, hence $\frac{d\Lambda}{d\theta} < 0$, and by (B.5),

$$\frac{\partial c_0(z, \theta)}{\partial \theta} > 0, \qquad \frac{\partial a(z, \theta)}{\partial \theta} < 0 \qquad \text{for all } \theta \in (0, 2).$$

□

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